HEC MONTRÉAL
École affiliée à l'Université de Montréal

Essays on Differential Games with Impulse Control

par
Utsav Sadana

Thèse présentée en vue de l'obtention du grade de Ph. D. en administration
(spécialisation Sciences de la décision)

Mars 2021

HEC MONTRÉAL

École affiliée à l'Université de Montréal

Cette thèse intitulée :

Essays on Differential Games with Impulse Control

Présentée par :

Utsav Sadana

a été évaluée par un jury composé des personnes suivantes :

Dena Firoozi
HEC Montréal
Présidente-rapportrice

Georges Zaccour
HEC Montréal
Directeur de recherche

Ngo Van Long
McGill University
Membre du jury

Herbert Dawid
Bielefeld University
Examinateur externe

Martin Cousineau
HEC Montréal
Représentant du directeur de HEC Montréal

Résumé

Cette thèse consiste en quatre essais sur des jeux différentiels déterministes à deux joueurs, à horizon fini et à somme non nulle, où un joueur implémente un contrôle continu pour influencer l'état, tandis que l'autre intervient à certains instants choisis stratégiquement. Le choix ne porte pas uniquement sur les moments d'interventions, mais aussi sur leurs niveaux. Les jeux dynamiques avec contrôles impulsionnels constituent une approche naturelle pour analyser le comportement stratégique des agents dans de nombreux contextes tels que l'investissement dans la qualité des produits, la réglementation environnementale et la cybersécurité. Cependant, la résolution de problèmes pratiques se heurte à une série de défis théoriques et computationnels, qui sont essentiellement dus à l'endogénéité des dates du contrôle impulsionnel. Ce travail relève certains de ces défis.

Dans cette thèse, nous caractérisons les équilibres de Nash sous les trois structures d'information qui ont été considérées dans la littérature sur la théorie des jeux différentiels, à savoir les structures d'information en boucle ouverte, en rétroaction et en données échantillonnées. De plus, nous montrons que la détermination des contrôles impulsionnels dans des jeux différentiels linéaires-quadratiques peut être obtenue comme solution d'un problème d'optimisation non linéaire sous contraintes. Dans le cas des jeux différentiels linéaires dans l'état, nous obtenons une caractérisation complètement analytique du nombre d'équilibres, le moment et l'amplitude des contrôles impulsionnels. Nous comparons aussi les résultats sous différentes structures d'information. Dans chaque essai, nous illustrons également les résultats théoriques en utilisant un jeu à deux joueurs, dont l'un préfère une valeur plus élevée de la variable d'état, tandis que l'autre vise à l'abaisser, une situation fréquente dans de nombreuses applications, en particulier en réglementation et en cybersécurité.

Mots-clés

Jeux différentiels, contrôles impulsionnels, équilibre de Nash en boucle ouverte, équilibre de Nash en rétroaction, équilibre de Nash à données échantillonnées, inégalités quasi-variationnelles, cybersécurité, réglementation

Méthodes de recherche

Théorie des jeux; programmation mathématique; analyse numérique

Abstract

This thesis consists of four essays on deterministic finite-horizon two-player nonzero-sum differential games where one player continuously controls the state while the other player strategically intervenes at certain (discrete) time instants to shift the state from one level to another. In contrast to classical differential games where all players take actions continuously, the impulse player also decides when to intervene during the game in addition to determining the level of the interventions. Impulse control models constitute a natural approach for analyzing strategic behavior of agents in many contexts such as investment in product quality, environmental regulation and cybersecurity. However, to solve practical problems, we need to address a series of theoretical and computational challenges that are due to the endogeneity of the timing of actions. This is the general topic of this work.

In this thesis, we characterize the Nash equilibria under all the three information structures that have been considered in the differential game theory literature, namely, open-loop, feedback and sampled-data information structures. Further, we show that the timing of impulses can be obtained as a solution of a constrained non-linear optimization problem in the case of linear-quadratic differential games with impulse controls. To analytically characterize the equilibrium number, timing, and magnitude of impulses, we introduce canonical linear-state game models and compare the equilibrium behavior of players across different information structures. In each essay, we also illustrate the theoretical results using a game problem between two players, one of whom prefers a higher state value while the other aims to lower the state, a situation that arises in many regulation and cybersecurity applications.

Keywords

Differential games, impulse control, open-loop Nash equilibrium, feedback Nash equilibrium, sampled-data Nash equilibrium, quasi-variational inequalities, cybersecurity, regulation

Research methods

Game theory; mathematical programming; numerical analysis

Contents

List of Tables

List of Figures

To my parents Aruna and Prem Kumar Sadana for their love and support which has made it possible

Acknowledgements

I am very lucky that Prof. Georges Zaccour is my advisor. Prof. Zaccour is one of the most kindest and supportive people that I have met in my life. His encouraging words and belief in me have allowed me to do research without any self doubt. Prof. Zaccour's insights on promising research directions have provided me opportunities to work on interesting and challenging problems. I will forever be grateful to Prof. Zaccour for his unwavering support which can be compared to that of my parents. I am also fortunate that I got a chance to be mentored by Prof. Puduru Viswanadha Reddy who guided me whenever I found myself lost. His research perspectives, knowledge, humility, and patience have contributed immensely to my growth as a researcher. I am very thankful to Vishwa and Prof. Zaccour for collaborating with me on all the chapters in this thesis and for their generosity and accessibility during my Ph.D. Both Prof. Zaccour and Vishwa have been instrumental in developing my interest in dynamic games.

I wish to thank Prof. Herbert Dawid, Prof. Michèle Breton, and Prof. Ngo van Long for being on my thesis committee. Prof. Breton and Prof. Long's encouragement, support, and guidance have helped me immensely in formulating the research problems in this thesis. I am grateful to Prof. Dawid for critically assessing my research as well for his encouragement and invaluable feedback. Prof. Dawid's comments have enabled me to think deeply about each chapter of this thesis and have also provided me with directions for future research.

I am grateful to Prof. Tamer Başar for hosting me at the Coordinated Science Lab (CSL) of the University of Illinois at Urbana-Champaign for my internship. The research environment at CSL is very inspiring and I spent some of the most memorable moments of my graduate studies at UIUC. I am also thankful to Prof. Başar for collaborating with me on the third chapter of his thesis and for sharing his knowledge on a range of topics in game theory. I am grateful to

Prof. Erick Delage for teaching me robust optimization and for his encouragement and generosity throughout my stay in Montréal.

I would also like to thank the following organizations for their financial support: Chair in Game Theory and Management, HEC Montréal, J.A. DeSève Foundation, Fonds québécois de la recherche sur la nature et les technologies (FRQNT), and Ministère de l'Éducation et de l'Enseignement supérieur du Québec (MEES).

This research could not have been possible without the love and constant support of my parents.

A huge thanks to Apala for her love and support. Special thanks to my friends for making my stay memorable in Montreal and UIUC. I am also thankful to the staff at GERAD and HEC Montréal who always tried to accommodate all my requests.

Preface

This thesis consists of four essays which are listed as follows:

1. Sadana, U., Reddy, P. V., and Zaccour, G. (2021). Nash equilibria in nonzero-sum differential games with impulse control. Accepted by European Journal of Operational Research.

2. Sadana, U., Reddy, P. V., and Zaccour, G. Open-loop and feedback Nash equilibria in scalar linear-state differential games with impulse control. Under third round review for publication in Automatica.

3. Sadana, U., Reddy, P. V., Başar, T., and Zaccour, G. Sampled-data Nash equilibria in differential games with impulse control. Under review for publication in Journal of Optimization Theory and Applications.

4. Sadana, U., Reddy, P. V., and Zaccour, G. Feedback Nash equilibria in differential games with impulse control. To be submitted.

Introduction

Game theory is a branch of mathematics that studies strategic interactions between intelligent and rational decision makers, called players. Strategic interactions take place anytime a player's payoff not only depends on her own decision, but also on the decisions made by the other players.

One-shot (or static) games are a useful representation of strategic interactions when the past and the future are irrelevant to the analysis, i.e., today's decisions only affect today's outcomes for the players and are independent of past moves. When there are carry-over effects and the players can condition their actions on history (and in particular on their rivals' behavior), then a dynamic game is needed. In a repeated game, the agents play the same game in each round, that is, the set of actions and the payoff structures are the same in all stages. The number of stages can be finite or infinite, and this distinction has been shown to have a tremendous impact on the equilibrium results. In multistage games, the players share the control of a discrete-time dynamic system (state equations) observed over stages. Their choice of control levels, e.g., investments in production capacity, or advertising dollars, affects the evolution of the state variables (e.g., production capacity, reputation of the firm), as well as current payoffs.

Differential games, which are the focus of this thesis, are continuous-time counterparts of multistage games. The literature on differential games typically assumes that all players take actions at each instant of time during the game, a setup that does not capture well many real-world applications where, for some reasons, one player only acts at some time instants. For instance, the production and marketing decisions are adjusted continuously by firms while changes to environmental (or other) tax policies are made at certain discrete time instants. Similarly, while a company builds continuously its infrastructure security system, a hacker attacks it only once in a while.

In all the aforementioned interactions, one player acts at each time instant during the game, while the other player intervenes only occasionally in the game. Since the number, timing, and level of the interventions (impulses) are decision variables of at least one of the players, these games are known as differential games with impulse controls, and provide a natural paradigm to model the interactions taking place in different contexts, namely: (i) law enforcement organizations deciding the impulse controls, that is, time of attack and resources to deploy, to disrupt the infrastructure of a terrorist organization that is continuously investing to build up its infrastructure; (ii) software firms investing in security to reduce the impact of a (potential) hacking attempt; (iii) regulators determining when and how much to change the emission taxes associated with pollutants; and (iv) governments deciding the timing and intensity (partial or complete) of lockdowns to control the spread of a virus.

Theoretical and computational developments for impulse optimal control problems that involve one agent have been extensively studied in the literature using the Pontryagin maximum principle (Blaquière, 1977a, 1979, 1985; Chahim et al., 2012, 2013) and Bensoussan-Lions quasi-variational inequalities (Bensoussan and Lions, 1982, 1984). Applications can be found in diverse fields, e.g., flood control (Chahim et al., 2013), forest management (Alvarez, 2004), cash management (Cadenillas and Zapatero, 1999; Bertola et al., 2016), epidemic models (Piunovskiy et al., 2020), cybersecurity (Taynitskiy et al., 2019), and product quality improvements (Reddy et al., 2016).

The literature on differential games with impulse controls is sparse and deals mostly with zero-sum games in options pricing (El Farouq et al., 2010) and pursuit evasion problems (Chikrii and Matichin, 2005; Chikrii et al., 2007). Recently, the authors in Aïd et al. (2020) and Ferrari and Koch (2019) introduced a class of two-player nonzero-sum stochastic impulse games where both players cause jumps to the state using their impulse controls only. In Basei et al. (2019), the authors extended these games to an N-player setting with $N > 2$, and also studied their mean-field counterpart. However, the players in these games have no continuous controls, which precludes the possibility of using them for situations where the interventions (hacking, change of emission taxes, lockdowns) by a player occur only at discrete instants of time and the state of the system (e.g., software vulnerabilities, pollution, infection rate) continuously evolves according to the actions (e.g., continuous effort in system security, production, social distancing) of another player.

To fill the gap in the literature, we introduce a general class of nonzero-sum differential games where one (representative) player uses piecewise-continuous controls to affect the continuously evolving state, while the other player intervenes using impulse controls to instantaneously change the state from one level to another. The discontinuities in state variable at endogenously determined impulse instants lead to computational difficulties in analyzing games with impulse controls. For tractability, Chang et al. (2013) and Chang and Wu (2015) studied them under the simplifying assumption that the impulse instants are given and the impulse player only selects the levels of impulses. Although this assumption holds for specific cases, e.g., a central bank changes interest rates at predetermined time instants during the year while the production and marketing decisions of the firms are made daily, there is no reason to believe that the timing of a government's attack on terrorist organizations or of a hacking attempt on a security firm is given a priori.

A central issue in the study of differential games is to determine the best way in which players can respond to one another. Using *Nash equilibrium* as a solution concept, the action profiles of both players can be determined for the whole duration of the game, where an equilibrium pair of strategies is such that no player has an incentive to unilaterally change their strategies.[1] The strategies are functions of the state information that is available to the players, and therefore, a change in the *information-structure* can affect the Nash equilibrium as well as the payoff of the players obtained in a differential game (Başar and Olsder, 1999).

The different information structures emerge due to the cost associated with state measurement, for instance, the economic data from the surveys can be obtained every quarter while firms make their production decisions daily. Therefore, the availability of state information can have policy implications for regulators interested in maximizing consumer welfare and government organizations protecting the citizens from terrorist attacks. Three kinds of information structure, namely, open-loop, feedback and sampled-data, have been predominantly studied in the literature. With open-loop information structure, players

[1] Each player's action profiles are generated using strategies that are mappings of the *information sets* (consisting of state measurements) to action sets (the set of admissible actions).

only know the initial state of the game, feedback information structure assumes that state can be measured at each time instant during the game, and state information is available at certain exogenously given sampling instants when the information structure is sampled-data.

This thesis contributes both to the theory and applications of differential games with impulse controls by providing a characterization of the Nash equilibrium under all the three information structures.

The first essay titled "Nash equilibria in nonzero-sum differential games with impulse control" introduces the general class of deterministic finite-horizon two-player nonzero-sum differential games where Player 1 uses piecewise-continuous controls whereas Player 2 uses impulse controls. The use of specialized controls for each player is motivated by applications in cybersecurity and regulation, and the more general case with both players using both piecewise continuous and impulse controls can be easily studied using our model. The characterization of open-loop Nash equilibrium can be shown to reduce to solving two coupled problems: (a) a non-standard optimal control problem of Player 1 with state jumps and additional costs at the impulse instants, (b) the impulse optimal control problem of Player 2. By extending the Pontryagin Maximum Principle to solve Player 1's non-standard optimal control and using the necessary optimality conditions for impulse control problems, we formulate the necessary and sufficient conditions for the existence of OLNE in the general class of nonzero-sum differential games with impulse controls.

The computation of equilibria is a hard problem as the impulse instants are fixed point solutions of a highly non-linear system of equations, known as the *Hamiltonian continuity condition* (Chahim et al., 2012), that are coupled with a system of differential equations. In the literature, infinite-horizon impulse games have been studied under an assumption that both players use impulses of the threshold type, that is, impulses occur only when the state enters an intervention set that is characterized by the quasi-variational inequalities (QVIs). The infinite-horizon assumption leads to time-independent intervention set (Aïd et al., 2020) which allows the authors to obtain closed-form solution for impulse games with linear and symmetric payoff functions.

We consider finite-horizon problems with open-loop information structure which do not impose structural assumptions on the impulse controls, and as a result, the impulses occur when both time and state values satisfy the Hamiltonian continuity condition. As a result, analytical solutions can be obtained for specific instances of differential games. Therefore, for the linear-quadratic differential games (LQDGs) that are widely studied in economics, management science and engineering, we obtain numerical solutions by providing, for a first time, a reformulation of the equilibrium conditions (coupled system of ordinary differential equations and Hamiltonian continuity condition) as a constrained non-linear optimization problem that can be solved by commercial optimization solvers. Even though, a priori, LQDGs appear restrictive as the payoff of players is assumed to be quadratic in state and the state dynamics are assumed to be linear in both state and controls of the players, one of the reasons for their ubiquity in optimal control literature is that the linear dynamics provide good approximations for the non-linear dynamics. Another advantage of using LQDGs is that they are tractable and at the same time, make it possible to account for state and control interactions, non-constant returns to scale and interactions between the players' control variables (Haurie et al., 2012; Başar and Olsder, 1999; Başar et al., 2018).

For analytical tractability, we consider linear-state differential games (LSDGs) which assume players' payoff functions to be linear in state (Dockner et al., 2000), and have been used in impulse games (see Aïd et al., 2020; Campi and De Santis, 2020). We show that there can be at most one interior equilibrium impulse in a LSDG and obtained equilibrium timing and level of impulse in closed-form. This is the first analytical characterization of a unique impulse in a nonzero-sum differential game. To illustrate the theory and algorithms developed in the first essay, a game between a government and an international terrorist organization (ITO) is formulated where the ITO invests efforts to builds its infrastructure that could be used later for an attack on civilians while the government launches strikes to disrupt ITO's resources. Previously, these problems were studied using classical dynamic game theory where all players take actions at all periods of the game (see Crettez and Hayek, 2014; Novak et al., 2010).

A classical result in deterministic linear-state differential games is that open-loop and feedback Nash equilibria coincide when all players make decisions at each time instant during the game (Dockner et al., 2000). This implies that players are not worse-off if they determine their actions using the state information at the initial time only. A natural question to answer is if this result holds in highly relevant problems in cybersecurity, terrorism and pollution regulation when one player uses impulse controls and the objective functions and state dynamics satisfy the LSDG formulation.

The objective of the second essay titled, "Open-loop and feedback Nash equilibria in scalar linear-state differential games with impulse controls" is to compare open-loop and feedback Nash equilibria obtained by using the Pontryagin Maximum Principle and quasi-variational inequalities, respectively, in deterministic LSDGs with impulse controls. We construct a canonical deterministic two-player LSDG of minimal configuration which allows us to include all the interactions between the players, and at the same time, keeps the analysis tractable. The objective functions of both players are linear in state and quadratic in controls, and without loss of generality, it is assumed that Player 1 uses piecewise-continuous controls and Player 2 uses impulse controls.

To assess the impact of impulse controls on the solution of a game under open-loop and feedback information structures, the following two situations are analyzed: First, the timing of impulses is considered to be exogenously given, and Player 2 determines the impulse levels at the corresponding impulse instants. In this case, open-loop and feedback information structures lead to the same equilibrium behavior of both players. Then, for the general case where the number and timing of impulses is determined by Player 2, it is shown that the classical result does not hold, that is, open-loop and feedback Nash equilibria are different. More specifically, OLNE has at most three impulses while FNE admits at most two impulses. Closed-form solutions for equilibrium timing of impulses and equilibrium strategies of both the players are obtained under OLNE and FNE. The differences in OLNE and FNE can be attributed to our result that impulse timing in OLNE depends on the problem parameters of Player 1 whereas the impulse timing in FNE depends only on Player 2's problem parameters and state dynamics. The results remain qualitatively unaltered for other general cost structures and the multi-dimensional extension of the scalar LSDG model. A numerical example is also provided to show that Player 2's intervention instant can be different depending on the information structure.

Open-loop and feedback information structures are the two extremes regarding the assumptions on the state information that is available to the players. It is well-known that open-loop strategies are only weakly time-consistent (Başar, 1989), and do not satisfy strong time-consistency which implies that at any time instant during the game, the players may have an incentive to unilaterally deviate from their equilibrium strategies. Even though feedback strategies are strongly time-consistent, they require state measurements to be made at each time instant which may not be feasible (Başar, 1989). A compromise is provided by the sampled-data information structure as state is measured at certain given sampling instants and sampled-data strategies are strongly time-consistent at the sampling instants. Sampled-data Nash equilibrium coincides with the open-loop Nash equilibrium of the game when sampling is done at the initial and final time only. An interesting problem then is to determine sampled-data Nash equilibrium for any given number of sampling instants (Simaan and Cruz Jr., 1973; Başar, 1991; Drăgan et al., 2019).

The third essay titled, "Sampled-data Nash equilibrium in differential games with impulse controls" studies the two-player game introduced in the first essay with the sampled-data information structure where the strategies of players depend on time and the last measured state value. We provide necessary and sufficient conditions for the existence of sampled-data Nash equilibria for the general class of differential games with impulse controls. An additional difficulty in computing sampled-data Nash equilibrium compared to OLNE is that the necessary conditions also include a Riccati system of equations for both the players. A reformulation of equilibrium conditions as a constrained non-linear optimization problem is provided for a scalar linear quadratic differential game, the solution of which gives the impulse instants.

For the class of LSDGs, the sampled-data Nash equilibrium is found to coincide with the open-loop Nash equilibrium. An extension of LSDG is also provided where the problem parameters vary over time and are constant between the sampling instants. In this case, the number of interventions can be at most equal to the number of sampling instants. A complete analytical characterization of equilibrium level of impulses and equilibrium controls of the players is also given for LSDGs. To show the contrast between differential games with periodic impulses and endogenous impulses, we consider a game where one player values the state positively and aims to increase it whereas the other player who values the state negatively invests efforts to lower it. Compared to the case when impulses are a priori assumed to be periodic, the equilibrium intervention instants occur at irregular intervals and the equilibrium controls of the players differ from the periodic case, thereby illustrating the need to include timing as a decision variable when studying equilibrium behavior of players in cybersecurity and regulation domains.

The control of exchange rate by the central bank of a country though direct interventions in the foreign exchange market and continuous control of interest rate is one of the most well-studied problems in impulse optimal control literature (Cadenillas and Zapatero, 1999; Bertola et al., 2016). This has also motivated the study of nonzero impulse games with feedback information structure where both players use only impulse control to keep the state close to their respective target values (Aïd et al., 2020). This game model with both players using impulse controls does not capture the interactions taking place between a firm that makes production decisions daily and a pollution regulator that intervenes at certain time instants to keep the pollution level close to their target value.

The fourth essay titled, "Feedback Nash equilibria in differential games with impulse controls" studies a general class of deterministic two-player finite-horizon nonzero-sum differential games with impulse controls assuming a feedback information structure. We show that the number of impulses in the game is finite and the Hamilton-Jacobi-Bellman equations coupled with a system of quasi-variational inequalities provide sufficient conditions to characterize the feedback Nash equilibrium. Further, we extend a well-known linear-quadratic impulse control problem to a deterministic LQDG problem in which the players incur costs if the state deviates from their target values. In a numerical example, it is shown that the equilibrium strategy of the impulse player is to intervene twice in the game. Our characterization of feedback Nash equilibrium in this essay is based on certain regularity assumptions on the value function that have been assumed in the literature (see, e.g., Campi and De Santis, 2020; Aïd et al., 2020). For the future work, we plan to relax these assumptions and develop policy iteration algorithms (Azimzadeh, 2019; Zabaljauregui, 2020) that can solve the quasi-variational inequalities for the impulse player.

References

Alvarez, L. H. R. (2004). Stochastic forest stand value and optimal timber harvesting. *SIAM Journal on Control and Optimization*, 42(6):1972–1993.

Azimzadeh, P. (2019). A zero-sum stochastic differential game with impulses, precommitment, and unrestricted cost functions. *Applied Mathematics & Optimization*, 79(2):483–514.

Aïd, R., Basei, M., Callegaro, G., Campi, L., and Vargiolu, T. (2020). Nonzero-sum stochastic differential games with impulse controls: A verification theorem with applications. *Mathematics of Operations Research*, 45(1):205–232.

Başar, T. (1991). Optimum $H\infty$ designs under sampled state measurements. *Systems & Control Letters*, 16(6):399 – 409.

Başar, T. (1995). Minimax control of switching systems under sampling. *Systems & Control Letters*, 25(5):315 – 325.

Başar, T. and Olsder, G. J. (1999). *Dynamic Noncooperative Game Theory, 2nd edition*. Society for Industrial and Applied Mathematics, Philadelphia, PA.

Başar, T. (1989). Time consistency and robustness of equilibria in noncooperative dynamic games. In der Ploeg, F. V. and de Zeeuw, A., editors, *Dynamic Policy Games in Economics*, pages 9 – 54. Springer International Publishing, North-Holland.

Başar, T., Haurie, A., and Zaccour, G. (2018). Nonzero-Sum Differential Games. In Başar, T. and Zaccour, G., editors, *Handbook of Dynamic Game Theory*, pages 61–110. Springer International Publishing, Cham.

Basei, M., Cao, H., and Guo, X. (2019). Nonzero-sum stochastic games and mean-field games with impulse controls.

Bensoussan, A. and Lions, J.-L. (1982). *Contrôle Impulsionnel et Inéquations Quasi-Variationelles*. Bordas, Paris, France.

Bensoussan, A. and Lions, J.-L. (1984). *Impulse Control and Quasi-Variational Inequalities*. Gauthier-Villars, Paris, France.

Bertola, G., Runggaldier, W. J., and Yasuda, K. (2016). On classical and restricted impulse stochastic control for the exchange rate. *Applied Mathematics & Optimization*, 74(2):423–454.

Blaquière, A. (1977a). Differential games with piece-wise continuous trajectories. In Hagedorn, P., Knobloch, H. W., and Olsder, G. J., editors, *Differential Games and Applications*, pages 34–69. Springer, Berlin.

Blaquière, A. (1979). Necessary and sufficient conditions for optimal strategies in impulsive control. In Lui, P. and Roxin, E., editors, *Differential Games and Control Theory III, Part A*, pages 1 – 28. Marcel Dekker, New York.

Blaquière, A. (1985). Impulsive optimal control with finite or infinite time horizon. *Journal of Optimization Theory and Applications*, 46(4):431–439.

Bokanowski, O., Maroso, S., and Zidani, H (2009). Some Convergence Results for Howard's Algorithm. journal = *SIAM Journal on Numerical Analysis*,47(4):3001-3026.

Cadenillas, A. and Zapatero, F. (1999). Optimal central bank intervention in the foreign exchange market. *Journal of Economic Theory*, 87(1):218 – 242.

Campi, L. and De Santis, D. (2020). Nonzero-sum stochastic differential games between an impulse controller and a stopper. *Journal of Optimization Theory and Applications*, 186(2):688–724.

Chahim, M. (2013). *Impulse control maximum principle: Theory and applications*. PhD thesis, Tilburg University.

Chahim, M., Brekelmans, R., den Hertog, D., and Kort, P. (2013). An impulse control approach to dike height optimization. *Optimization Methods and Software*, 28(3):458–477.

Chahim, M., Hartl, R. F., and Kort, P. M. (2012). A tutorial on the deterministic Impulse Control Maximum Principle: Necessary and sufficient optimality conditions. *European Journal of Operational Research*, 219(1):18–26.

Chang, D., Wang, H., and Wu, Z. (2013). Maximum principle for non-zero sum differential games of BSDEs involving impulse controls. In *Proceedings of the 32nd Chinese Control Conference*, pages 1564–1569.

Chang, D. and Wu, Z. (2015). Stochastic maximum principle for non-zero sum differential games of FBSDEs with impulse controls and its application to finance. *Journal of Industrial & Management Optimization*, 11(1):27–40.

Chikrii, A. A. and Matichin, I. (2005). Linear differential games with impulse control of players. *Proceedings of the Steklov Institute of Mathematics*, Suppl. 1:S68–S81.

Chikrii, A. A., Matychyn, I. I., and Chikrii, K. A. (2007). Differential games with impulse control. In Jørgensen, S., Quincampoix, M., and Vincent, T. L., editors, *Advances in Dynamic Game Theory. Annals of the International Society of Dynamic Games*, volume 9, pages 37–55. Birkhäuser, Boston.

Crettez, B. and Hayek, N. (2014). Terrorists' eradication versus perpetual terror war. *Journal of Optimization Theory and Applications*, 160:679–702.

Dockner, E. J., Jorgensen, S., Long, N. V., and Sorger, G. (2000). *Differential Games in Economics and Management Science*. Cambridge University Press.

Drăgan, V., Ivanov, I. G., and Popa, I.-L. (2019). Stochastic linear quadratic differential games in a state feedback setting with sampled measurements. *Systems & Control Letters*, 134:104563.

El Farouq, N., Barles, G., and Bernhard, P. (2010). Deterministic minimax impulse control. *Applied Mathematics and Optimization*, 61(3):353 – 378.

Ferrari, G. and Koch, T. (2019). On a strategic model of pollution control. *Annals of Operations Research*, 275(2):297–319.

Haurie, A., Krawczyk, J. B., and Zaccour, G. (2012). *Games and Dynamic Games*. World Scientific, Singapore.

Novak, A., Feichtinger, G., and Leitmann, G. (2010). A Differential Game Related to Terrorism: Nash and Stackelberg Strategies. *Journal of Optimization Theory and Applications*, 144:533–555.

Piunovskiy, A., Plakhov, A., and Tumanov, M. (2020). Optimal impulse control of a sir epidemic. *Optimal Control Applications and Methods*, 41(2):448–468.

Reddy, P. V., Wrzaczek, S., and Zaccour, G. (2016). Quality effects in different advertising models - An impulse control approach. *European Journal of Operational Research*, 255(3):984–995.

Simaan, M. and Cruz Jr., J. (1973). Sampled-data Nash controls in non-zero-sum differential games. *International Journal of Control*, 17(6):1201 – 1209.

Taynitskiy, V., Gubar, E., and Zhu, Q. (2019). Optimal Impulse Control of SIR Epidemics Over Scale-Free Networks. In Song, J. B., Li, H., and Coupechoux, M., editors, *Game Theory for Networking Applications*, pages 119–129. Springer International Publishing, Cham.

Yong, J. (1994). Zero-sum differential games involving impulse controls. *Applied Mathematics and Optimization*, 29(3):243–261.

Zabaljauregui, D. (2020). A fixed-point policy-iteration-type algorithm for symmetric nonzero-sum stochastic impulse control games. *Applied Mathematics & Optimization*.

Chapter 1

Nash equilibria in nonzero-sum differential games with impulse control

Abstract

In this paper, we introduce a class of deterministic finite-horizon two-player nonzero-sum differential games where one player uses ordinary[1] controls while the other player uses impulse controls. We formulate the necessary and sufficient conditions for the existence of an open-loop Nash equilibrium for this class of differential games. We specialize these results to linear-quadratic games, and show that the open-loop Nash equilibrium strategies can be computed by solving a constrained non-linear optimization problem. In particular, for the impulse player, the equilibrium timing and level of impulses can be obtained. Furthermore, for the special case of linear-state differential games, we obtain analytical characterization of equilibrium number, timing and the level of impulse in terms of the problem data. We illustrate our results using numerical experiments.

[1]We use the word 'ordinary' to mean that Player 1 uses control strategies that are piecewise continuous functions of time.

1.1 Introduction

In this paper, we consider dynamic competitive strategic situations involving two players, one of whom takes actions only occasionally, while the other makes decisions continuously. One example of such a setting is a central bank that announces its interest rate policy at specific dates during the year, while firms make production and marketing decisions daily. Another example is in cybersecurity, where an attacker launches its viruses to inflict damage on a system at strategic instants of time, while the defender is continuously investing in reducing the system's vulnerability. Each of the interventions (or impulses) by the central bank or the hacker can cause a jump in the state variable and additional terms in the objectives of the players. The two examples, which can be modeled as finite-horizon differential games with one impulse player, differ in terms of one crucial feature. In the first case, the impulse player (the central bank) states in advance when interest rate announcements will be made.[2] The pending decision is then the impulse size, that is, the interest rate itself (or the change relative to its current value). In the second case, both the timing of the impulses and their values are endogenous, and quite naturally, no one expects the hacker to give the defender advance notice of when the attacks will take place. Intuitively, solving for the cybersecurity game equilibrium is harder than determining the equilibrium strategies in the central bank game.

The contribution of the paper is four fold. *First*, we introduce a canonical two-player nonzero-sum differential game where one player uses ordinary controls and the other player uses impulse controls. We emphasize that our model is canonical in nature, that is, ordinary and impulsive decision variables are attributed to Player 1 and Player 2, respectively. The general case where both players are endowed with both types of control variables can be studied easily as an extension of the current framework. *Second*, we derive necessary and sufficient conditions for the existence of an open-loop Nash equilibrium for this class of games. *Third*, we specialize our results to linear-quadratic setting, and provide a reformulation of the equilibrium conditions as a constrained non-linear optimization problem for numerically computing the open-loop Nash equilibrium. Applications of linear-quadratic differential games (LQDGs) have been popular

[2]For instance, the Bank of Canada's interest rate announcements are available on https://www.bankofcanada.ca/press/upcoming-events/.

for decades in economics, engineering, and management science. One reason for this is the availability of theorems characterizing the existence and uniqueness of Nash and Stackelberg equilibria (see, e.g., Başar and Olsder 1999; Engwerda 2005; Haurie et al. 2012; Başar et al. 2018). Another reason is that, notwithstanding the specific functional forms of the objectives, LQDGs make it possible to account for three features that are usually important in these applications, namely, interactions between the players' control variables, interactions between the control and state variables, and finally non-constant returns to scale. *Fourth*, for the class of linear-state games, we show that, for the player who uses impulse controls, the equilibrium strategy has at most one impulse, and analytically characterize the time and the level of impulse in terms of the problem data.

By establishing existence results for the class of two-player differential games with one impulse player, and providing solution methods for their applications in LQDGs, our paper contributes to both the theory and applications of differential games. Surprisingly, the literature on differential games with impulse control is very sparse. In such a context, as has happened in the past, one takes stock on what has been achieved in the two sister areas, namely, optimal control and zero-sum differential games.

The rest of the paper is organized as follows. In Section 1.1.1, we briefly review the relevant literature on optimal control and differential games with impulse controls. In Section 1.2, we introduce our model. In Section 1.3, we extend the Pontryagin maximum principle to optimal control problems with additional discrete state cost terms and state jumps. Using this result, we provide necessary and sufficient conditions for the existence of open-loop Nash equilibrium. In Section 1.4, we specialize these results to linear-quadratic differential games. More specifically, in Section 1.4.2, we present an algorithm to determine the equilibrium timing and level of the impulses for Player 2. In Section 1.5, we provide the analytical solution of the linear-state differential game. In Section 1.6, we present numerical illustration of the results. Concluding remarks are given in Section 1.7.

1.1.1 Literature review

In this section, we first review the literature on impulse optimal control problems and zero-sum differential games with impulse controls. Next, we present the advances in the study of nonzero-sum impulse games where all players use impulse controls, which allows us to contrast our work on nonzero-sum differential games where one player uses piecewise continuous controls while the other player uses impulse controls.

A number of variants of impulse optimal control problems have been studied in the literature. A series of papers have considered the case where the number of jump instants is fixed (Liu et al., 1998; Wu and Teo, 2006) or the impulse instants are known a priori (Taynitskiy et al., 2019; Reddy et al., 2016). Impulse control problems are typically solved using the maximum principle provided by Blaquière (1977a,b, 1979, 1985). Papers where dynamic programming is used to compute the solutions of impulse control problems include Neuman and Costanza (1990), Erdlenbruch et al. (2013), and Bertola et al. (2016). However, analytical solutions could not be derived from the maximum principle. Consequently, a number of algorithms, such as the gradient method and a continuation method based on formulating a multi-point boundary value problem, have been proposed in the literature to numerically compute the solutions (see, e.g., Grass and Chahim 2012; Chahim 2013; Grames et al. 2019; Kort 1989; Hou and Wong 2011).

Reddy et al. (2016) extended some well-known advertising models by adding impulse investments in quality. In Chahim et al. (2017), a firm decides on the timing of the adoption of a new technology as well as the level of investments in new capital at the corresponding instants. In these two papers, the optimal solutions are computed by formulating a multi-point boundary value problem, which generalizes the two-point boundary value problem to account for the additional restrictions on the state dynamics and co-state variable at the interior impulse instants. In Chahim et al. (2013), the authors determined the optimal timing and corresponding dike heightenings to protect against floods. In Erdlenbruch et al. (2013), the authors studied a renewable resource-management problem with an impulse controlled harvesting policy, which is a sequence of harvest times and harvesting levels of the resource. Recently, in the field of cybersecurity, Taynitskiy et al. (2019) introduced discrete time periodic patching processes in the continuous-time Susceptible-Infected-Recovered (SIR) epidemic model to control the malware's spread in devices. Impulse control problems have been extensively studied in management because they allow for discrete time interventions in continuous time processes, see, e.g., Eastham and Hastings (1988), Chahim et al. (2012), Bensoussan et al. (2012), Bertola et al. (2016), Basei (2019), Perera et al. (2020).

The literature on differential games with impulse controls is sparse, and a majority of the existing works consider a zero-sum setting. In Chikrii et al. (2007), the sufficient conditions for hitting a target set are provided for a pursuit-evasion game where either the pursuer or the evader or both can give a finite number of impulses to the system. The pursuer's objective is to make the state trajectory hit a target set in finite time while the evader aims to steer the system trajectory away from the target for as long as possible. Bernhard et al. (2006) and El Farouq et al. (2010) introduced impulse control in zero-sum differential games to study an option pricing problem. Zero-sum impulse control differentiable games with one player using piecewise continuous controls and the other using impulse controls are studied in a deterministic setting in Yong (1994) and in a stochastic setting in Zhang (2011) and Azimzadeh (2019). Recent works that use dynamic programming to determine the equilibrium include Cosso (2013), El Asri and Mazid (2018), Ferrari and Koch (2019), Azimzadeh (2019), Aïd et al. (2020), and Campi and De Santis (2020). Stochastic differential games where both players use only impulse control are considered in Cosso (2013) and El Asri and Mazid (2018) for the zero-sum case, whereas Aïd et al. (2020) and Ferrari and Koch (2019) studied the nonzero-sum case for an infinite horizon problem. In the pollution regulation problem studied in Ferrari and Koch (2019), both the regulator and polluting firm use impulse controls. Campi and De Santis (2020) analyze a two-player nonzero-sum differential game where one player uses impulse controls while the other player can stop the game at any time.

In order to study the differential game between two nations that have different targets for the currency exchange rate, Aïd et al. (2020) provide a system of quasi-variational inequalities (Bensoussan and Lions, 1982, 1984) that need to be solved in order to compute the Nash equilibrium. To the best of our knowledge, Aïd et al. (2020) is the only other paper in the literature that has provided analytical solutions for Nash equilibria in linear-state games with impulse controls. However, they have assumed a symmetric linear-state game to determine the analytical solution and obtained multiple equilibria. It is to be noted that they do not allow for piecewise continuous controls in their model. The N-player extension of the impulse game in Aïd et al. (2020) is studied in Basei et al. (2019)

where the authors provide conditions for the existence of an ϵ-Nash equilibrium, and analyze its mean-field counterpart. Chang et al. (2013) and Chang and Wu (2015) have used the maximum principle to deal with a nonzero-sum stochastic differential game with impulse controls in a finite-horizon setting. However, they assume that the timing of impulse is given, and players only choose the level of impulse. From the above discussion, we can clearly see that driven by different applications in options pricing, currency exchange rate or regulation, the differential game problems with impulse are specialized in the kind of controls that are available to the players.

1.2 Model

In this section, we introduce a class of finite-horizon nonzero-sum two-player differential game models where one player uses ordinary controls whereas the other player uses impulse controls. Let $T < \infty$ be the duration of the game. The control action of Player 1 at time $t \in [0, T]$ is denoted by $u(t) \in \Omega_u$, where Ω_u is the action set of Player 1 which is a subset of $\mathbb{R}^{m_1}$. Here, $u : [0, T] \to \Omega_u$ is assumed to be a piecewise continuous function of time and denotes the strategy of Player 1. The set of strategy profiles of Player 1 is denoted by $\mathcal{U}$. Player 2 intervenes or takes actions only at certain isolated time instants (or impulse instants) during the time period $[0, T]$. We denote the set of intervention instants of Player 2 by $\{\tau_1, \tau_2, \cdots, \tau_k\},\ k \in \mathbb{N}$, where $\mathbb{N}$ is the set of natural numbers. The impulse instants satisfy the following monotone increasing sequence property,

$$0 \leq \tau_1 < \tau_2 < \cdots < \tau_k \leq T. \tag{1.1}$$

At each time instant τ_i $(i = 1, 2, \cdots, k)$, Player 2 takes an action or uses the control $v_i \in \Omega_v$, where Ω_v is the action set of Player 2 which is a subset of $\mathbb{R}^{m_2}$. A strategy of Player 2 is denoted by $\tilde{v} = \big(\big\{(\tau_1, v_1), (\tau_2, v_2), \cdots, (\tau_k, v_k)\big\}, k\big) \in \mathcal{V}$, where $\mathcal{V}$ is the strategy set of Player 2. We note that the number of impulses k, the level or size of the impulse v_i $(i = 1, 2, \cdots, k)$, and the timing of impulses τ_i $(i = 1, 2, \cdots, k)$ are decision variables of Player 2.

Using their control variables, the players influence the evolution of the system or the interaction environment as follows:

$$\dot{x}(t) = f(x(t), u(t)),\ x(0^-) = x_0,\ \text{for } t \neq \{\tau_1, \tau_2, ..., \tau_k\}, \tag{1.2a}$$

$$x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i^-), v_i), \text{ for } i = \{1, 2, ..., k\}, \tag{1.2b}$$

where $x(t) \in \mathbb{R}^n$ denotes the state of the system at time $t \in [0, T]$ and 0^- denotes the time instant just before 0. The state variable just before and after the impulse instant τ_i is given by $x(\tau_i^-) = \lim_{t \uparrow \tau_i} x(t)$ and $x(\tau_i^+) = \lim_{t \downarrow \tau_i} x(t)$, respectively. The initial state $x_0 \in \mathbb{R}^n$ is assumed to be given. The objectives of the players are described as follows: Player 1 uses a strategy $u(.) \in \mathcal{U}$ to maximize the objective

$$J_1(x_0, u(.), \tilde{v}) = \int_0^T F_1(x(t), u(t))dt + \sum_{i=1}^{k} G_1(x(\tau_i^-), v_i) + S_1(x(T^+)), \tag{1.3a}$$

and Player 2 uses a strategy $\tilde{v} \in \mathcal{V}$ to maximize the objective

$$J_2(x_0, u(.), \tilde{v}) = \int_0^T F_2(x(t), u(t))dt + \sum_{i=1}^{k} G_2(x(\tau_i^-), v_i) + S_2(x(T^+)), \tag{1.3b}$$

where F_j denotes the running payoff of Player j, G_j is the additional cost incurred by Player j at the impulse instants, S_j represents the terminal payoff (salvage value) of Player j, and T^+ denotes the time instant just after T. We have the following assumptions regarding the system in (1.2) and the objectives in (1.3).

Assumption 1.1 *(a) The function $f : \mathbb{R}^n \times \Omega_u \to \mathbb{R}$ is Lipschitz continuous in x for all u such that, for $c > 0$, we have*

$$|f(x, u) - f(y, u)| \le c|x - y|.$$

(b) The functions $F_j : \mathbb{R}^n \times \Omega_u \to \mathbb{R}$, $G_j : \mathbb{R}^n \times \Omega_v \to \mathbb{R}$, $j = 1, 2$, and $g : \mathbb{R}^n \times \Omega_v \to \mathbb{R}$ are jointly continuous in their arguments, and have continuous partial derivatives with respect to their arguments. The terminal payoff functions $S_j : \mathbb{R}^n \to \mathbb{R}$, $j = 1, 2$, are continuous and have continuous partial derivatives with respect to their arguments.

(c) The action sets of the players Ω_u and Ω_v are compact and convex subsets of $\mathbb{R}^{m_1}$ and $\mathbb{R}^{m_2}$, respectively.

(d) The number of impulse actions used by Player 2 is bounded, that is, there exists a natural number $N < \infty$ such that $k < N$.

(e) For τ_i to be an admissible impulse instant, the corresponding impulse level v_i should be such that $g(\cdot, v_i) \neq 0$.

Assumptions 1.(a) and 1.(c) ensure that there exists a unique state trajectory $x(\cdot)$ for any measurable $u(\cdot)$ and impulse sequence $\{(\tau_i, v_i)\}$, $i = \{1, 2, \cdots, k\}$. Assumptions 1.(b)-1.(d) are common in applied differential games (see, e.g., Haurie et al. 2012; Başar et al. 2018) and impulse optimal control theory (Geering, 1976; Chahim et al., 2013, 2017). Assumption 1.(e) ensures that there are no degenerate impulse instants for which the corresponding jumps in state are of size equal to zero.

We seek to find Nash equilibrium strategies for the differential game defined by (1.2a)-(1.3b).

Definition 1.1 *The strategy profile* $(u^*(.), \tilde{v}^*)$ *is a Nash equilibrium of the differential game* (1.2a)-(1.3b) *if the following inequalities hold true:*

$$J_1(x_0, u^*(.), \tilde{v}^*) \geq J_1(x_0, u(.), \tilde{v}^*), \ \forall u(.) \in \mathcal{U}, \tag{1.4a}$$

$$J_2(x_0, u^*(.), \tilde{v}^*) \geq J_2(x_0, u^*(.), \tilde{v}), \ \forall \tilde{v} \in \mathcal{V}. \tag{1.4b}$$

It is well-known that, in a differential game, the Nash equilibrium varies with the adopted information structure, that is, the information that the players use when making their decisions; see Başar and Olsder (1999). In an open-loop information structure, the players' strategies only depend on time t (and initial state x_0, which is a given parameter). In closed-loop and feedback information structures, the players' strategies depend on time and the state variable. As a first step in dealing with a non-zero-sum differential game with an impulse player, we adopt the simplest information structure, that is, open-loop.

Remark 1.1 *The two-player dynamic game described by* (1.2a)-(1.3b) *is canonical in nature, that is, the minimal configuration to analyze the interaction of ordinary and impulsive controls. An extension to multiple players, with players using both ordinary and impulse controls, follows directly from the current framework.*

Remark 1.2 *In Yong (1994) and El Farouq et al. (2010), the authors study zero-sum differential games with impulse controls. In these works, it is assumed that players restrict their strategies to non-anticipative strategies (see Elliott et al., 1972), as the main objective is to obtain Markov perfect or feedback Nash equilibrium. In this paper, we assume an open-loop information structure, where players commit to using the entire strategy, which is a description of actions defined over the time horizon* $[0, T]$. *As a result, we do not further restrict the strategy spaces beyond the description provided in the model.*

1.3 Open-loop Nash equilibrium

In this section, we derive the necessary and sufficient conditions for the existence of an open-loop Nash equilibrium for the differential game (1.2a)-(1.3b).

1.3.1 Necessary conditions

We show that the Nash equilibrium conditions (1.4) result in a system of weakly coupled optimal control problems. First, if the strategy profile $(u^*(.), \tilde{v}^*)$ is a Nash equilibrium, then $u^*(.)$ is the best response to Player 2's strategy $\tilde{v}^* := (\{(\tau_1^*, v_1^*), (\tau_2^*, v_2^*), \cdots, (\tau_{k^*}^*, v_{k^*}^*)\}, k^*)$, that is, $\tilde{u}^*$ solves the following optimal control problem for Player 1:

$$\max_{u(.)\in\mathcal{U}} \quad J_1(x_0, u(.), \tilde{v}^*), \tag{1.5a}$$

$$\text{subject to} \quad \dot{x}(t) = f(x(t), u(t)), \qquad \forall t \neq \{\tau_1^*, \tau_2^*, \cdots, \tau_{k^*}^*\}, \tag{1.5b}$$

$$x(\tau_i^{*+}) = x(\tau_i^{*-}) + g(x(\tau_i^{*-}), v_i^*), \qquad \forall i = \{1, 2, \cdots, k^*\}, \tag{1.5c}$$

where $x(0^-) = x_0$ and

$$J_1(x_0, u(.), \tilde{v}^*) = \int_0^T F_1(x(t), u(t))dt + \sum_{i=1}^{k^*} G_1(x(\tau_i^{*-}), v_i^*) + S_1(x(T^+)).$$

Remark 1.3 *Player 2's strategy $\tilde{v}^*$ induces jumps in the state variable in* (1.5c) *and additional payoffs $G_1(x(\tau_i^{*-}), v_i^*)$ in* (1.5a). *This implies that the above problem differs from a classical optimal control problem due to the presence of additional payoffs $G_1(x(\tau_i^{*-}), v_i^*)$ as well as the jumps in the state variable at specific instants of time $\{\tau_1^*, \tau_2^*, \cdots, \tau_{k^*}^*\}$ given by* (1.5c).

Optimal control problems where the objective function has additional terms added at specific instants of time have been studied in the literature; see Geering (1976) and Getz and Martin (1980). In Player 1's optimal control problem (1.5), there exist jumps in the state variable besides the additional terms in the objective. Further, to analyze the influence of Player 2's equilibrium strategy, we provide an auxiliary result related to the necessary conditions for optimality associated with the optimal control problem in (1.5).

Theorem 1.1 (Extended Pontryagin Principle for state jumps) *Let Assumption 1.1 hold true. Let* $(x^*(.), u^*(.))$ *be an optimal solution of Player 1's problem* (1.5). *Then, there exists a piecewise continuous and piecewise differentiable co-state trajectory* $\lambda_1(.)$, *with* $\lambda_1(t) \in \mathbb{R}^n$, *such that for* $t \notin \{\tau_1^*, \tau_2^*, \cdots, \tau_{k^*}^*\}$, *the Hamiltonian function is given by*

$$H_1(x(t), u(t), \lambda_1(t)) := F_1(x(t), u(t)) + \lambda_1(t)^T f(x(t), u(t)), \tag{1.6a}$$

the optimal control satisfies

$$u^*(t) = \arg \max_{u(t) \in \Omega_u} H_1(x^*(t), u(t), \lambda_1(t)), \tag{1.6b}$$

the maximized Hamiltonian is defined as

$$H_1^*(x^*(t), \lambda_1(t)) = H_1(x^*(t), u^*(t), \lambda_1(t)), \tag{1.6c}$$

the state and co-state variables satisfy

$$\dot{x}^*(t) = H_{1\lambda_1}^*(x^*(t), \lambda_1(t)),\ x^*(0^-) = x_0, \tag{1.6d}$$

$$\dot{\lambda}_1(t) = -H_{1x}^*(x^*(t), \lambda_1(t)),\ \lambda_1(T^+) = S_{1x}(x^*(T^+)). \tag{1.6e}$$

At the impulse instants $\{\tau_1^*, \tau_2^*, \cdots, \tau_{k^*}^*\}$, *the following jump conditions hold true:*

$$x^*(\tau_i^{*+}) = x^*(\tau_i^{*-}) + g(x^*(\tau_i^{*-}), v_i^*), \tag{1.6f}$$

$$\lambda_1(\tau_i^{*-}) = (I + (g_x(x^*(\tau_i^{*-}), v_i^*))^T)\lambda_1(\tau_i^{*+}) + G_{1x}(x^*(\tau_i^{*-}), v_i^*). \tag{1.6g}$$

Proof. See Appendix 1.8.1 ■

The jumps in the state and co-state in (1.6f) and (1.6g), respectively, are induced by Player 2's equilibrium impulse strategy.

Remark 1.4 *Several extensions of the classical maximum principle are available in the optimal control literature; see Seierstad and Sydsæter (1987). Some of these extensions are related to hybrid dynamics and jumps in the state variable; see Sussmann (1999) and Getz and Martin (1980). To analyze the effect of Player 2's impulsive actions on Player 1's optimal behavior, pertinent to our canonical model, we provide the auxiliary result in Theorem 1.1 using the needle variation approach.*

Next, we consider Player 2's problem (1.4b), given Player 1's equilibrium strategy $u^*(.)$. Then, $\tilde{v}^*$ is the best response of Player 2 if it solves the following impulse optimal control problem:

$$\begin{aligned}
&\max_{\tilde{v}\in\mathcal{V}} \quad J_2(x_0, u^*(.), \tilde{v}), && && (1.7a)\\
&\text{subject to} \quad \dot{x}(t) = f(x(t), u^*(t)), && \forall t \neq \{\tau_1, \tau_2, \cdots, \tau_k\}, && (1.7b)\\
&\qquad x(\tau_i^+) = x(\tau_i^-) + g(x(\tau_i^-), v_i), && \forall i = \{1, 2, \cdots, k\}, && (1.7c)
\end{aligned}$$

where $x(0^-) = x_0$ and

$$J_2(x_0, u^*(.), \tilde{v}) = \int_0^T F_2(x(t), u^*(t))dt + \sum_{i=1}^{k} G_2(x(\tau_i^-), v_i) + S_2(x(T^+)).$$

The Hamiltonian[3] and impulse Hamiltonian associated with the above impulse optimal control problem are defined, respectively, by

$$H_2(x(t), u^*(t), \lambda_2(t)) := F_2(x(t), u^*(t)) + \lambda_2(t)^T f(x(t), u^*(t)) \tag{1.8}$$

$$H_2^I(x(t), v_i, \lambda_2(t)) := G_2(x(t), v_i) + \lambda_2(t)^T g(x(t), v_i). \tag{1.9}$$

Notice that in our canonical model, Player 2 uses only impulse controls, and does not affect the vector field (1.7b). In the following theorem using the auxiliary result of Theorem 1.1, we formulate the necessary conditions for the existence of an open-loop Nash equilibrium. The proof of the theorem uses the necessary conditions associated with an impulse optimal control problem (see Chahim et al., 2012, Theorem 1).

Theorem 1.2 (Necessary conditions) *Let Assumption 1.1 hold true. Let* $(u^*(.), \tilde{v}^*)$ *be an open-loop Nash equilibrium of the differential game described by* (1.2)-(1.3). *Then, there exist piecewise continuous and piecewise differentiable functions* $\lambda_1(.)$ *and* $\lambda_2(.)$ *with* $\lambda_1(t) \in \mathbb{R}^n$ *and* $\lambda_2(t) \in \mathbb{R}^n$ *such that the following conditions hold for* $t \in [0, T]$ *: for* $t \notin \{\tau_1^*, \tau_2^*, ..., \tau_{k^*}^*\}$, *the equilibrium control of Player 1 satisfies*

$$u^*(t) = \arg\max_{u\in\Omega_u} H_1(x^*(t), u(t), \lambda_1(t)), \tag{1.10a}$$

the maximized Hamiltonian and impulse Hamiltonian functions are given by

$$H_1^*(x^*(t), \lambda_1(t)) = H_1(x^*(t), u^*(t), \lambda_1(t)), \tag{1.10b}$$

[3]Player 1's equilibrium strategy $u^*(.)$ influences the vector field (1.7b). Hence, $u^*(t)$ appears in Player 2's Hamiltonian.

$$H_2^{I^*}(x^*(\tau_i^{*-}), \lambda_2(\tau_i^{*+})) = H_2^I(x^*(\tau_i^{*-}), v_i^*, \lambda_2(\tau_i^{*+})), \tag{1.10c}$$

the equilibrium state and co-state equations admit

$$\dot{x}^*(t) = f(x^*(t), u^*(t)),\ x^*(0^-) = x_0, \tag{1.10d}$$

$$\dot{\lambda}_1(t) = -H_{1x}^*(x^*(t), \lambda_1(t)),\ \lambda_1(T^+) = S_{1x}(x^*(T^+)), \tag{1.10e}$$

$$\dot{\lambda}_2(t) = -H_{2x}^*(x^*(t), u^*(t), \lambda_2(t)),\ \lambda_2(T^+) = S_{2x}(x^*(T^+)). \tag{1.10f}$$

At the impulse instant τ_i^* ($i = 1, 2, \cdots, k$), the equilibrium control of Player 2 satisfies

$$v_i^* = \arg\max_{v_i \in \Omega_v} H_2^I(x^*(\tau_i^{*-}), v_i, \lambda_2(\tau_i^{*+})), \tag{1.10g}$$

the jumps in the state and co-state variables satisfy

$$x^*(\tau_i^{*+}) = x^*(\tau_i^{*-}) + g(x^*(\tau_i^{*-}), v_i^*), \tag{1.10h}$$

$$\lambda_1(\tau_i^{*-}) = (I + (g_x(x^*(\tau_i^{*-}), v_i^*))^T)\lambda_1(\tau_i^{*+}) + G_{1x}(x^*(\tau_i^{*-}), v_i^*), \tag{1.10i}$$

$$\lambda_2(\tau_i^{*-}) = \lambda_2(\tau_i^{*+}) + H_{2x}^{I\ *}(x^*(\tau_i^{*-}), \lambda_2(\tau_i^{*+})), \tag{1.10j}$$

and the following Hamiltonian consistency condition holds true:

$$H_2(x^*(\tau_i^{*+}), u^*(\tau_i^{*+}), \lambda_2(\tau_i^{*+})) - H_2(x^*(\tau_i^{*-}), u^*(\tau_i^{*-}), \lambda_2(\tau_i^{*-})) \begin{cases} > 0 & \text{for } \tau_i^* = 0 \\ = 0 & \text{for } \tau_i^* \in (0, T) \\ < 0 & \text{for } \tau_i^* = T \end{cases}. \tag{1.10k}$$

Proof. See Appendix 1.8.2. ■

Remark 1.5 *When impulse instants are interior, that is $\tau_i^* \in (0, T)$ ($i = 1, 2, \cdots, k^*$), we have that Hamiltonian of Player 2 is continuous at the impulse instant τ_i^*, that is, the necessary condition* (1.10k) *holds with equality.*

Remark 1.6 *For our finite-horizon game, with open-loop information structure, the impulse instants are characterized by the Hamiltonian continuity condition. That is, the time and state values for which the condition* (1.10k) *holds true. This is analogous to the intervention set which characterizes the impulse instants with feedback information structure. An intervention set is also defined in terms of time and state values; see*

Bertola et al. (2016). However, for the feedback information structure assumed in Aïd et al. (2020), an impulse occurs when the state leaves a time-independent continuation set. This is due to the fact that the authors have considered an infinite-horizon impulse game and the impulse controls are assumed, a priori, to be of the threshold-type, that is, interventions occur only when the state leaves the continuation set that is characterized by the QVIs. We have not made any structural assumption of this kind on the impulse controls.

1.3.2 Sufficient conditions

In this subsection, we provide conditions under which the necessary conditions (1.10) are also sufficient. Suppose the strategy profile $(u^*(.), \tilde{v}^*)$ is obtained by solving (1.10). To show that the necessary conditions (1.10) are also sufficient, we have to show that $u^*(.)$ is a best response to $\tilde{v}^*$, that is an optimal solution for the problem (1.4a), and $\tilde{v}^*$ is a best response to $u^*(.)$, that is an optimal solution for the problem (1.4b). In the next theorem, we provide the required sufficient conditions. The proof uses sufficient conditions for optimality associated with the impulse optimal control problem (1.4b); see Chahim et al. (2012) and Seierstad (1981, Theorem 1).

Assumption 1.2 *The impulse instants given by* (1.1) *are interior, that is, $\tau_i \in (0, T)$ for $i = 1, 2, \cdots, k$.*

Theorem 1.3 (Sufficient conditions) *Let Assumptions 1.1 and 1.2 hold. Suppose there exist feasible solutions $(u^*(.), \tilde{v}^*)$, state trajectory $x^*(.)$ and co-state trajectories $\lambda_1(.)$ and $\lambda_2(.)$, such that the conditions given by* (1.10) *are satisfied. Then $(u^*(.), \tilde{v}^*)$ is an open-loop Nash equilibrium of the differential game described by* (1.2)-(1.3) *if the following conditions hold:*

(i) the maximized Hamiltonian $H_1^*(x(t), \lambda_1(t))$ of Player 1 is concave in $x(t)$ for all $\lambda_1(t)$,

(ii) the Hamiltonian $H_2(x(t), u^*(t), \lambda_2(t))$ of Player 2 is concave in $x(t)$,

(iii) the salvage values $S_1(x(T))$ and $S_2(x(T))$ are concave in $x(T)$,

(iv) $G_1(x(t), v) + \lambda_1^T g(x(t), v)$ is concave in $x(t)$,

(v) the impulse Hamiltonian $H_2^I(x(t), v, \lambda_2(t))$ *of Player 2 is concave in* $(x(t), v)$.

Proof. See Appendix 1.8.3. ■

1.4 Linear-quadratic differential game with impulse control

In this section, we specialize the obtained results to linear-quadratic differential games, and provide an algorithm for computing the open-loop Nash equilibrium. We introduce the following two-player linear quadratic version of the differential game (1.2)-(1.3). This new game will be referred to as iLQDG (where the i stands for impulse):

$$\text{iLQDG}: \qquad \max_{u(.)\in\mathcal{U}} J_1(x_0, u(.), \tilde{v}),\ \max_{\tilde{v}\in\mathcal{V}} J_2(x_0, u(.), \tilde{v}), \tag{1.11a}$$

$$\text{subject to} \quad \dot{x}(t) = Ax(t) + Bu(t),\ \forall t \neq \{\tau_1, \tau_2, \cdots, \tau_k\}, \tag{1.11b}$$

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i,\ \forall i = \{1, 2, \cdots, k\}, \tag{1.11c}$$

where $x(0^-) = x_0$ and the players' objectives are given by

$$\begin{aligned} J_j(x_0, u(.), \tilde{v}) = &\int_0^T \frac{1}{2}\big(x(t)^T W_j x(t) + 2w_j^T x(t) + u(t)^T R_j u(t) + 2d_j^T u(t)\big)dt \\ &+ \sum_{i=1}^k \frac{1}{2}\big(x(\tau_i^-)^T Z_j x(\tau_i^-) + 2q_j^T x(\tau_i^-) + v_i^T P_j v_i + 2p_j^T v_i\big) \\ &+ \frac{1}{2}(x(T^+)^T S_j x(T^+) + 2s_j^T x(T^+)),\ j = 1, 2, \end{aligned} \tag{1.11d}$$

where $R_j \in \mathbb{R}^{m_1\times m_1},\ P_j \in \mathbb{R}^{m_2\times m_2}$, $W_j \in \mathbb{R}^{n\times n}$, $Z_j \in \mathbb{R}^{n\times n}$, $S_j \in \mathbb{R}^{n\times n},\ j = 1, 2.$

Remark 1.7 *To keep the presentation parsimonious, we omit cross terms between the state and control variables. If the objective functions include such terms, then by using suitable transformations, they can be reduced (Engwerda, 2005, pp. 100) to the canonical form given in* (1.11d)*; see Section 1.6 for an illustration.*

Assumption 1.3 *We assume that*

1. *The matrices* W_j, Z_j, S_j, $j = 1, 2$, R_2 *and* P_1 *are symmetric, and the matrices* R_1 *and* P_2 *are symmetric and negative definite.*

2. *Player 1's strategy space $\mathcal{U}$ is the set of locally square-integrable functions, that is,*

$$\mathcal{U} := \left\{ u(t) \in \mathbb{R}^{m_1},\ t \in [0, T] \;\middle|\; \int_0^T u^T(t) u(t) dt < \infty \right\}, \tag{1.12}$$

and the strategy of Player 2 is given by

$$\mathcal{V} := \Big\{ \{(\tau_1, v_1), (\tau_2, v_2), \cdots, (\tau_k, v_k), k\},\ k \in \mathbb{N} \;\Big|\; v_i \in \Omega_v,\ \tau_i \in (0, T),$$
$$i = 1, 2, \cdots, k,\ 0 < \tau_1 < \tau_2 < \cdots < \tau_k < T \Big\}. \tag{1.13}$$

3. *The number of impulse actions used by Player 2 is bounded, that is $k < N$ for some $N < \infty$ and the impulse instants satisfy Assumption 1.2.*

1.4.1 Necessary conditions

In this subsection, we provide necessary conditions for an open-loop Nash equilibrium associated with iLQDG. For later use and simplification, we introduce some additional notation. The equilibrium state and co-state variables are arranged as a column vector $y(t) \in \mathbb{R}^{3n}$ such that $y(t) := [x(t)^T\ \lambda_1(t)^T\ \lambda_2(t)^T]^T$ for all $t \in [0, T]$. To describe the evolution of equilibrium state and co-state variables, we introduce the following $3n \times 3n$ matrices:

$$M := \begin{bmatrix} A & -BR_1^{-1}B^T & 0 \\ -W_1 & -A^T & 0 \\ -W_2 & 0 & -A^T \end{bmatrix},\ \eta_1 := \begin{bmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\ \eta_2 := \begin{bmatrix} 0 & 0 & 0 \\ -S_1 & I & 0 \\ -S_2 & 0 & I \end{bmatrix}, \tag{1.14a}$$

$$N := \begin{bmatrix} I + QP_2^{-1}Q^T Z_2 & 0 & -QP_2^{-1}Q^T \\ -Z_1 & I & 0 \\ -Z_2 & 0 & I \end{bmatrix}. \tag{1.14b}$$

The Hamiltonian function associated with Player 2 is calculated as

$$H_2(x(t), \lambda_1(t), \lambda_2(t)) = \frac{1}{2} x(t)^T W_2 x(t) + (w_2 + A^T \lambda_2(t))^T x(t) + \frac{1}{2}(R_2 R_1^{-1} B^T \lambda_1(t)$$
$$- 2B^T \lambda_2(t) + R_2 R_1^{-1} d_1 - 2d_2)^T R_1^{-1}(B^T \lambda_1(t) + d_1). \tag{1.15}$$

Remark 1.8 *The elementary operation of premultiplying the third block row of the matrix N with $QP_2^{-1}Q^T$ and addition with the first block row results in a lower triangular matrix with diagonal elements equal to 1. This implies that the matrix N is invertible.*

In the next theorem, we state the necessary conditions for open-loop Nash equilibrium associated with the iLQDG.

Theorem 1.4 (Necessary conditions) *Let Assumption 1.3 hold true. Let $(u^*(.), \tilde{v}^*)$ be an open-loop Nash equilibrium of iLQDG. Then, there exist piecewise continuous and piecewise differentiable functions $\lambda_1(.)$ and $\lambda_2(.)$ with $\lambda_1(t) \in \mathbb{R}^n$ and $\lambda_2(t) \in \mathbb{R}^n$, such that the following conditions hold for $t \in [0, T]$:*
for $t \neq \{\tau_1^*, \tau_2^*, \cdots, \tau_{k^*}^*\}$

$$u^*(t) = -R_1^{-1}(B^T\lambda_1(t) + d_1), \tag{1.16a}$$

$$\dot{y}(t) = My(t) + C, \tag{1.16b}$$

$$\eta_1 y(0) + \eta_2 y(T) = X_0, \tag{1.16c}$$

and at the impulse instants τ_i^* ($i = 1, 2, \cdots, k^*$), Player 2's control and jump in $y(\tau_i^*)$ satisfy

$$v_i^* = -P_2^{-1}(Q^T\lambda_2(\tau_i^{*+}) + p_2), \tag{1.16d}$$

$$y(\tau_i^{*+}) = Ny(\tau_i^{*-}) + K, \tag{1.16e}$$

and the following Hamiltonian continuity condition holds true

$$H_2(x(\tau_i^{*+}), \lambda_1(\tau_i^{*+}), \lambda_2(\tau_i^{*+})) = H_2(x(\tau_i^{*-}), \lambda_1(\tau_i^{*-}), \lambda_2(\tau_i^{*-})), \tag{1.16f}$$

where $y(t) = [x(t)^T\ \lambda_1(t)^T\ \lambda_2(t)^T]^T$, $C = -[(BR_1^{-1}d_1)^T\ w_1^T\ w_2^T]^T$, $K = [(Q^Tq_2 - p_2)^T P_2^{-1}Q^T\ - q_1^T\ - q_2^T]^T$, and $X_0 = [x_0^T\ s_1^T\ s_2^T]^T$.

Proof. See Appendix 1.8.4 ■

1.4.2 Solvability

In this subsection, under a few additional assumptions on the problem data, we show that the solution of the necessary conditions (1.16) can be reformulated as a solution of a constrained non-linear optimization problem. First, from the state equation (1.16b), $y(\tau_1^{*-})$ is calculated as

$$y(\tau_1^{*-}) = \phi(\tau_1^{*-}, 0)y(0) + \varphi(\tau_1^{*-}, 0),$$

where $\phi(\tau_1^{*-},0) = e^{M\tau_1^{*-}}$ and $\varphi(\tau_1^{*-},0) = e^{M\tau_1^{*-}} \int_0^{\tau_1^{*-}} e^{-Ms}Cds$. Next, given operators $\phi(\tau_i^{*-},0)$ and $\varphi(\tau_i^{*-},0)$ for $i > 1$, $y(\tau_i^{*-})$ and $y(\tau_{i+1}^{*-})$ can be determined as follows:

$$y(\tau_i^{*-}) = \phi(\tau_i^{*-},0)y(0) + \varphi(\tau_i^{*-},0), \tag{1.17a}$$

$$y(\tau_{i+1}^{*-}) = \phi(\tau_{i+1}^{*-},0)y(0) + \varphi(\tau_{i+1}^{*-},0). \tag{1.17b}$$

Using (1.16b)-(1.16e), we have

$$\begin{aligned} y(\tau_{i+1}^{*-}) &= e^{M(\tau_{i+1}^{*-}-\tau_i^{*+})}y(\tau_i^{*+}) + e^{M\tau_{i+1}^{*-}}\int_{\tau_i^{*+}}^{\tau_{i+1}^{*-}} e^{-Ms}Cds, \\ &= e^{M(\tau_{i+1}^{*-}-\tau_i^{*+})}(Ny(\tau_i^{*-}) + K) + e^{M\tau_{i+1}^{*-}}\int_{\tau_i^{*+}}^{\tau_{i+1}^{*-}} e^{-Ms}Cds, \\ &= e^{M(\tau_{i+1}^{*-}-\tau_i^{*+})}\left(N\left(\phi(\tau_i^{*-},0)y(0) + \varphi(\tau_i^{*-},0)\right) + K\right) + e^{M\tau_{i+1}^{*-}}\int_{\tau_i^{*+}}^{\tau_{i+1}^{*-}} e^{-Ms}Cds, \\ &= e^{M(\tau_{i+1}^{*-}-\tau_i^{*+})}N\phi(\tau_i^{*-},0)y(0) + e^{M(\tau_{i+1}^{*-}-\tau_i^{*+})}\left(N\varphi(\tau_i^{*-},0) + K\right) \\ &\quad + e^{M\tau_{i+1}^{*-}}\int_{\tau_i^{*+}}^{\tau_{i+1}^{*-}} e^{-Ms}Cds. \end{aligned}$$

Comparing the above equations with (1.17b), we can compute $\phi(\tau_j^{*-},0)$ and $\varphi(\tau_j^{*-},0)$ recursively for $j \in \{1,2,\cdots,k^*\}$ as follows:

$$\phi(\tau_{j+1}^{*-},0) = e^{M(\tau_{j+1}^{*-}-\tau_j^{*+})}N\phi(\tau_j^{*-},0), \tag{1.18a}$$

$$\varphi(\tau_{j+1}^{*-},0) = e^{M(\tau_{j+1}^{*-}-\tau_j^{*+})}(N\varphi(\tau_j^{*-},0) + K) + e^{M\tau_{j+1}^{*-}}\int_{\tau_j^{*+}}^{\tau_{j+1}^{*-}} e^{-Ms}Cds, \tag{1.18b}$$

$$\text{with } \phi(\tau_1^{*-},0) = e^{M\tau_1^{*-}},\ \varphi(\tau_1^{*-},0) = e^{M\tau_1^{*-}}\int_0^{\tau_1^{*-}} e^{-Ms}Cds, \text{ and } \tau_{k^*+1}^* = T. \tag{1.18c}$$

Player 2's Hamiltonian function (1.15) can be written, after a few algebraic manipulations, in the following quadratic form:

$$H_2(y(t)) = \frac{1}{2}y(t)^T\mathbf{A_1}y(t) + \mathbf{b}^Ty(t) + \mathbf{c}, \tag{1.19}$$

where $\mathbf{A_1} := \begin{bmatrix} W_1 & 0 & 0 \\ 0 & BR_1^{-1}R_2^TR_1^{-1}B^T & 0 \\ 2A & -2BR_1^{-1}B^T & 0 \end{bmatrix}$, $\mathbf{b} := \begin{bmatrix} w_1 \\ BR_1^{-1}(R_2R_1^{-1}d_1 - d_2) \\ -BR_1^{-1}B^Td_1 \end{bmatrix}$, and $\mathbf{c} := \frac{1}{2}(R_2R_1^{-1}d_1 - 2d_2)^TR_1^{-1}B^Td_1$. We have the following theorem concerning the solvability of the necessary conditions (1.16).

Theorem 1.5 (Solvability) *Let Assumption 1.3 hold true. Let* $(u^*(.), \tilde{v}^*)$ *be an open-loop Nash equilibrium of the linear-quadratic game described by* (1.11). *Let* $\{\tau_1^*, \tau_2^*, \cdots, \tau_{k^*}^*\}$ *be the interior equilibrium impulse instants. Let the matrices* $\phi(\tau_i^{*-}, 0)$ *and* $\varphi(\tau_i^{*-}, 0)$ *for* $i \in \{1, 2, \cdots, k^*\}$ *be calculated recursively using* (1.18), *and let the matrix* $(\eta_1 + \eta_2 \phi(T, 0))$ *be invertible. Then, the state-co-state vectors* $y(t) = [x(t)^T, \lambda_1(t)^T, \lambda_2(t)^T]^T$ *for* $t \in (\tau_i^*, \tau_{i+1}^*)$ *and* $i \in \{1, 2, \cdots, k^*\}$ *are solved as*

$$y(t) = e^{M(t-\tau_{i^*}^+)} N\phi(\tau_i^{*-}, 0)y(0) + e^{M(t-\tau_{i^*}^+)}(N\varphi(\tau_i^{*-}, 0) + K) + e^{Mt} \int_{\tau_i^{*+}}^{t} e^{-Ms} C \, ds, \tag{1.20a}$$

with $y(0) = (\eta_1 + \eta_2 \, \phi(T, 0))^{-1}(X_0 - \eta_2 \varphi(T, 0))$ *and* $\tau_{k^*+1}^* = T$. *Further, the state-co-state vector satisfies the following quadratic equality constraint at the switching instant:* τ_i^*, $i \in \{1, 2, \cdots, k^*\}$:

$$\frac{1}{2} y(\tau_i^{*-})^T (N^T \mathbf{A_1} N - \mathbf{A_1}) y(\tau_i^{*-}) + \big(\frac{1}{2} K^T (\mathbf{A_1}^T + \mathbf{A_1}) N + \mathbf{b}^T N - \mathbf{b}^T\big) y(\tau_i^{*-}) + \frac{1}{2} K^T \mathbf{A_1} K + \mathbf{b}^T K = 0. \tag{1.20b}$$

The open-loop equilibrium strategies of the players are given by

$$u^*(t) = -R_1^{-1}(B^T[0 \; I \; 0] y(t) + d_1), \text{ for } t \in [0, T] \backslash \{\tau_1^*, \tau_2^*, \cdots \tau_{k^*}^*\}, \tag{1.20c}$$

$$v_i^* = -P_2^{-1}\Big(Q^T[0 \; 0 \; I]\big(N\phi(\tau_i^*, 0)y(0) + (N\varphi(\tau_i^*, 0) + K)\big) + p_2\Big), \quad \text{for } i \in \{1, 2, \cdots, k^*\}. \tag{1.20d}$$

Proof. See Appendix 1.8.5. ■

Remark 1.9 *In general, there may exist multiple equilibria for an iLQDG. When the number of impulses and the timing of the impulse instants are exogenously given, that is, they are not decision variables,*[4] (1.20a), *along with* (1.20c) *and* (1.20d) *provides the unique open-loop Nash equilibrium. In Theorem 1.5, though the impulse instants are not known, the number of impulse instants* k^* *must be specified a priori.*[5]

[4]In the central bank example mentioned in the introduction, the timing of change in the interest rate is exogenous.

[5]This approach is usually followed in the area of impulse optimal control, see, Chahim et al. (2013) and the references therein. In Chahim (2013) and Grass and Chahim (2012), the authors compute the impulse optimal control by first fixing the number of impulse instants a priori, and then later choose the number that maximizes the objective.

Non-linear programming formulation

Following Assumption 1.1.(d) on the bound on the number of impulses, the equilibrium number of impulses is obtained by first determining the equilibrium payoff of Player 2 for a fixed number of impulses, and then selecting the number of impulses that maximize the payoff of Player 2. To illustrate this observation, we denote by $\mathcal{V}^k$ the strategy set where Player 2 gives exactly k non-degenerate impulses, that is,

$$\mathcal{V}^k := \Big\{ \{(\tau_1, v_1), (\tau_2, v_2), \cdots, (\tau_k, v_k)\} \;\Big|\; v_i \in \Omega_v,\ \tau_i \in (0, T),\ i = 1, 2, \cdots, k, \\ 0 < \tau_1 < \tau_2 < \cdots < \tau_k < T \Big\}.$$

We denote a strategy in the set $\mathcal{V}^k$ by $\tilde{v}_k := \{(\tau_1, v_1), (\tau_2, v_2), \cdots, (\tau_k, v_k)\}$. Due to Assumption 1.1.(d), there exists $N < \infty$ such that the strategy space $\mathcal{V}$ of Player 2 can be partitioned as $\mathcal{V} := \mathcal{V}^1 \cup \mathcal{V}^2 \cup \cdots \mathcal{V}^N$ where $\mathcal{V}^i \cap \mathcal{V}^j = \emptyset$ for $i \neq j$ and $i, j \in \{1, 2, \cdots, N\}$. Using this, and from (1.4b), Player 2's optimization problem can be written as

$$J_2(u^*(.), \tilde{v}^*) = \max_{\tilde{v} \in \mathcal{V}} J_2(u^*(.), \tilde{v}) = \max_{k \in \{1,2,\cdots,N\}} \max_{\tilde{v}_k \in \mathcal{V}^k} J_2(u^*(.), \tilde{v}_k). \tag{1.21}$$

Let $\tilde{v}_k^*$ denote the optimal solution of the problem $\max_{\tilde{v}_k \in \mathcal{V}^k} J_2(u^*(.), \tilde{v}_k)$, then we have

$$k^* = \operatorname*{argmax}_{k \in \{1,2,\cdots,N\}} J_2(u^*(.), \tilde{v}_k^*) \text{ and } \tilde{v}^* = \{\tilde{v}_{k^*}^*, k^*\}. \tag{1.22}$$

From (1.22), it is evident that the equilibrium number of impulse instants is obtained by first fixing the number of impulse instants and solving the inner optimization problem in (1.21), and then solving the outer optimization problem (1.22). Using this observation, we provide a non-linear programming based approach for solving the necessary conditions (1.20).

For a given number of impulses k, the equilibrium impulse instants $\boldsymbol{\tau} := \{\tau_1^*, \tau_2^*, \cdots, \tau_k^*\}$ are characterized by the Hamiltonian continuity condition (1.20b) and $y(\tau_i^*)$ can be computed from the recursive relations (1.17a) and (1.18). (1.20b) is a non-linear function of impulse instants so it is difficult to determine τ_i^* analytically. However, we can determine the impulse instants τ_i^* numerically by solving a constrained non-linear optimization problem. We describe the procedure as follows.

In Assumption 1.3, we have supposed separability of the the impulse instants, i.e.,

$$\tau_1 < \tau_2 < \cdots < \tau_k,$$

and this constraint can be represented as

$$D\boldsymbol{\tau} < \mathbf{0}, \tag{1.23}$$

where

$$D := \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}_{(k-1)\times k}, \quad \boldsymbol{\tau} := \begin{bmatrix} \tau_1 \\ \vdots \\ \tau_k \end{bmatrix}.$$

The strict inequality (1.23) can be transformed as an inequality by introducing a negative `slack` variable[6] so that $D\boldsymbol{\tau} \leq \texttt{slack}$. Next, again from Assumption 1.3, the impulses cannot occur at the initial and final time. This can be ensured by defining the lower and upper bounds on each impulse instant as $\texttt{lb} \leq \boldsymbol{\tau} \leq \texttt{ub}$, where $\texttt{lb} = -\texttt{slack}$ and $\texttt{ub} = T + \texttt{slack}$. Player 2 can choose any negative value of the slack variable such that the interior impulse instants satisfy the separability condition in Assumption 1.3. The impulse instants associated with the strategy $\tilde{v}_k^*$ are then computed by solving the following constrained minimization problem:

$$\begin{aligned} \boldsymbol{\tau}^* = \underset{\texttt{lb}\leq\boldsymbol{\tau}\leq\texttt{ub}}{\operatorname{argmin}} \quad & \texttt{obj}(\boldsymbol{\tau}) \\ \text{subject to} \quad & D\boldsymbol{\tau} \leq \texttt{slack}, \end{aligned} \tag{1.24}$$

where

$$\begin{aligned} \texttt{obj}(\boldsymbol{\tau}) = \sum_{i=1}^{k} \Big(& \frac{1}{2} y(\tau_i^*)^T (N^T \mathbf{A_1} N - \mathbf{A_1}) y(\tau_i^*) \\ & + (\frac{1}{2} K^T (\mathbf{A_1}^T + \mathbf{A_1}) N + \mathbf{b}^T N - \mathbf{b}^T) y(\tau_i^*) + \frac{1}{2} K^T \mathbf{A_1} K + \mathbf{b}^T K \Big)^2. \end{aligned}$$

The non-linear optimization problems can be solved by using an interior-point algorithm (Waltz et al., 2006; Byrd et al., 1999) or sequential quadratic programming (SQP) methods (see Powell, 1978; Barclay et al., 1998; Büskens and Maurer, 2000). In this paper, we use the non-linear programming solver `fmincon` in MATLAB for solving (1.24) numerically.

[6] Usually, slack variables are used to transform an inequality constraint into an equality constraint. We use a slack variable to transform a strict inequality constraint to an inequality constraint.

1.4.3 Sufficient conditions

Theorem 1.5 provides a way of solving the necessary conditions (1.16), and as a result, the obtained solutions are candidates for the open-loop Nash equilibrium. In the next theorem, we provide conditions under which these candidates are indeed the open-loop Nash equilibrium solutions. The proof of the theorem directly follows from the sufficient conditions stated in Theorem 1.3.

Theorem 1.6 (Sufficient conditions) *Let Assumption 1.3 hold true. Let the matrices* $\{W_i, Z_i, S_i, i = 1, 2\}$ *be symmetric and negative semi-definite, and the matrices* $\{R_1, P_2\}$ *be symmetric and negative definite. Further, let the matrix* $(\eta_1 + \eta_2\phi(T, 0))$ *be invertible, Then, the solutions* $(u^*(.), \tilde{v}^*)$ *given by* (1.20c) *and* (1.20d) *provide the open-loop Nash equilibrium strategies of iLQDG described by* (1.11).

1.5 Linear-state differential game with impulse control

In this subsection, we specialize our results to one-dimensional linear-state differential games to obtain sharper results concerning the existence of open-loop Nash equilibrium strategies. The players maximize their objective functions given by

$$J_1(x_0, u(.), \tilde{v}) = \int_0^T \frac{1}{2}\left(2w_1 x(t) + R_1 u^2(t)\right) dt + \sum_{i=1}^{k} q_1 x(\tau_i^-) + s_1 x(T^+), \tag{1.25a}$$

$$J_2(x_0, u(.), \tilde{v}) = \int_0^T w_2 x(t) dt + \sum_{i=1}^{k} \frac{1}{2} P_2 v_i^2 + s_2 x(T^+), \tag{1.25b}$$

with the evolution of the state given by

$$\dot{x}(t) = Ax(t) + Bu(t),\ x(0^-) = x_0,\ \text{for}\, t \neq \{\tau_1, \tau_2, ..., \tau_k\}, \tag{1.25c}$$

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i,\ \text{for}\, i = \{1, 2, ..., k\}. \tag{1.25d}$$

In the next theorem, we will show that for the linear-state game, described by (1.25), it is possible to obtain, from the problem data, an analytical characterization of timing and level of impulse for Player 2's equilibrium strategy.

Theorem 1.7 *Consider the linear-state differential game described by* (1.25)*. Let Assumption 1.3 hold true. In equilibrium, the number of impulse instants for Player 2 is at most one, that is,* $k^* \leq 1$*. In particular, if the parameters satisfy the following conditions:*

$$A \neq 0, \ \frac{Q^2 w_2}{P_2} \neq \frac{B^2 q_1}{R_1}, \tag{1.26a}$$

$$T + \frac{1}{A} \ln \left(\left(\frac{Q}{B} \right)^2 \frac{R_1}{P_2} \frac{A s_2 + w_2}{q_1} \right) > 0, \tag{1.26b}$$

$$\frac{1}{A} \ln \left(\left(\frac{Q}{B} \right)^2 \frac{R_1}{P_2} \frac{A s_2 + w_2}{q_1} \right) < 0, \tag{1.26c}$$

then $k^* = 1$*. Further, the associated impulse level and the impulse timing are given by*

$$\tau^* = T + \frac{1}{A} \ln \left(\left(\frac{Q}{B} \right)^2 \frac{R_1}{P_2} \frac{A s_2 + w_2}{q_1} \right), \tag{1.27}$$

$$v^* = \frac{Q w_2}{P_2 A} - \frac{B^2 q_1}{A Q R_1}. \tag{1.28}$$

Proof. See Appendix 1.8.6 ■

In the above theorem, we have used the fact that the co-state of Player 2 is strictly monotone in order to show that the impulse instant is unique, and there can be at most one interior impulse. To the best of our knowledge, this is the first analytical characterization of a unique equilibrium in a specialized differential game with impulse control. It can be clearly seen that both the equilibrium timing and level of impulse depend on the problem parameters that appear in the objective function of Player 1. If the ratio of Q and B is kept constant, then the size of impulse at equilibrium can be altered without affecting the equilibrium timing of impulse.

1.6 Numerical illustration

In this section, we provide a numerical illustration of our results for iLQDGs.

Consider a dynamic game between a government and an international terrorist organization (ITO) where the ITO continuously builds its resources that can be financial assets or infrastructure to plan attacks while the government carries out strikes to disrupt the ITO's resources. In the literature (see Novak et al., 2010; Crettez and Hayek, 2014) on security applications, open-loop Nash equilibrium solutions have been derived to study these strategic interactions under the assumption that both the government and ITO act at each instant of time in the game. However, in practice, the strikes are carried out by the government at certain discrete instants of time. We model the aforementioned interaction using a finite-horizon differential game with impulse controls where the government decides the number and timing of its strikes besides the optimal effort level to be invested in the strike. The numerical values for the dynamic game between ITO and government have been chosen to illustrate that the theory and algorithms developed in this paper also apply to iLQDGs involving control-state interactions. For future work, it would be interesting to apply our dynamic game using real-world data.

At any time $t \in [0, T]$, let $x(t)$ denote the resources of ITO. Clearly, the government's running payoff decreases with increase in the ITO's resources while the running payoff of the ITO increases as they build more resources. However, there is a cost of building resources which is a quadratic function of the effort level, $u(t)$, of the ITO, and the cost of strike for the government is quadratic in effort level v_i of the government. At the time of the strike, the ITO incurs a loss of $\frac{3}{2}x(\tau_i^-)^2$, which clearly increases with ITO's resources at the time of the attack. At the end of the horizon which is assumed to be 25, the government incurs a loss if the terrorist resources are not destroyed completely while terminal reward of ITO is increasing in its resources for $x < 2$.

The objectives of ITO and government are given by

$$\begin{aligned} J_I(x_0, u(.), \tilde{v}) =& \frac{1}{2}\Big(\int_0^{25} [-5x(t)^2 - 4u(t)^2 + 8x(t)u(t) + 4x(t)]dt \\ & - \sum_{i=1}^{k} \big(3x(\tau_i^-)^2\big) - x(25^+)^2 + 4x(25^+)\Big), \\ J_G(x_0, u(.), \tilde{v}) =& \frac{1}{2}\Big(\int_0^{25} -x(t)^2 dt - \sum_{i=1}^{k} 2v_i^2 - 5x(25^+)^2\Big), \end{aligned}$$

and the state dynamics by

$$\begin{aligned} &\dot{x}(t) = -0.2x(t) + 0.2u(t),\ x(0^-) = 1,\ \text{for}\, t \neq \{\tau_1, \tau_2, ..., \tau_k\}, \\ &x(\tau_i^+) = x(\tau_i^-) - 0.3v_i,\ \text{for}\ i = \{1, 2, ..., k\}, \end{aligned}$$

where there is natural depreciation of resources given by $-0.2x(t)$. The state increases with increase in the continuous effort level $u(t)$ of the ITO and decreases with effort v_i invested in the strike by the government. Note that ITO's objective includes the control-state cross term $8x(t)u(t)$. We can write

$$-5x(t)^2 - 4u(t)^2 + 8x(t)u(t) + 4x(t) = -4(u(t) - x(t))^2 - x(t)^2 + 4x(t),$$

and obtain an equivalent iLQDG by making the substitution $\bar{u}(t) = u(t) - x(t)$. Upon rewriting the objective functions, we have

$$\begin{aligned} J_I(x_0, \tilde{u}(.), \tilde{v}) =& \frac{1}{2}\Big(\int_0^{25} [-x(t)^2 + 4x(t) - 4\bar{u}^2(t)]dt - \sum_{i=1}^{k} \big(3x(\tau_i^-)^2\big) - x(25^+)^2 \\ & + 4x(25^+)\Big), \\ J_G(x_0, \tilde{u}(.), \tilde{v}) =& \frac{1}{2}\Big(\int_0^{25} -x(t)^2 dt - \sum_{i=1}^{k} 2v_i^2 - 5x(25^+)^2\Big), \end{aligned}$$

and the state dynamics are given by

$$\begin{aligned} &\dot{x}(t) = 0.2\bar{u}(t),\ x(0^-) = 1,\ \text{for}\, t \neq \{\tau_1, \tau_2, ..., \tau_k\}, \\ &x(\tau_i^+) = x(\tau_i^-) - 0.3v_i,\ \text{for}\, i = \{1, 2, ..., k\}. \end{aligned}$$

We use the non-linear optimization based procedure given in Section 1.4.2 for calculating the equilibrium instants at which government strikes the ITO. Recall, from Remark 1.9, that the number of impulse instants must be specified a priori in order to use the solver. The timing and level of strike by the government are shown in Table 1.1 for different number of strikes. Clearly, government's strategy for any number of strikes greater than one is to attack the ITO with effort levels that decrease monotonically over time.

Table 1.1 – Equilibrium timing and level of impulses for different exogenous numbers of impulses

k	(τ_i, v_i)	J_I	J_G
0	-	41.89	-38.27
1	(6.86, 4.43)	-28.89	-39.74
2	(6.1, 3.79), (12.07, 2.9)	9.53	-36.5
3	(5.48, 3.39), (11.53, 2.41), (12.24, 2.32)	1.73	-34.90
4	(4.95, 3.11) (11.30 ,2.10) (12.09 , 2.00) (18.48, 1.37)	10.12	-31.34

We can see from Figure 1.1 that compared with the game with no impulses, the government receives lower payoff by giving one impulse and the payoff of the government increases for $k \leq 4$. Our numerical experiments show that there are no impulses for $k > 4$. So, the equilibrium number of impulse instants can be taken as 4. However, there can be multiple equilibria for a given number of impulses.

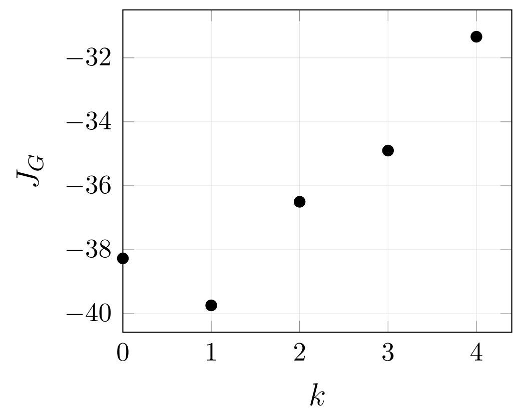
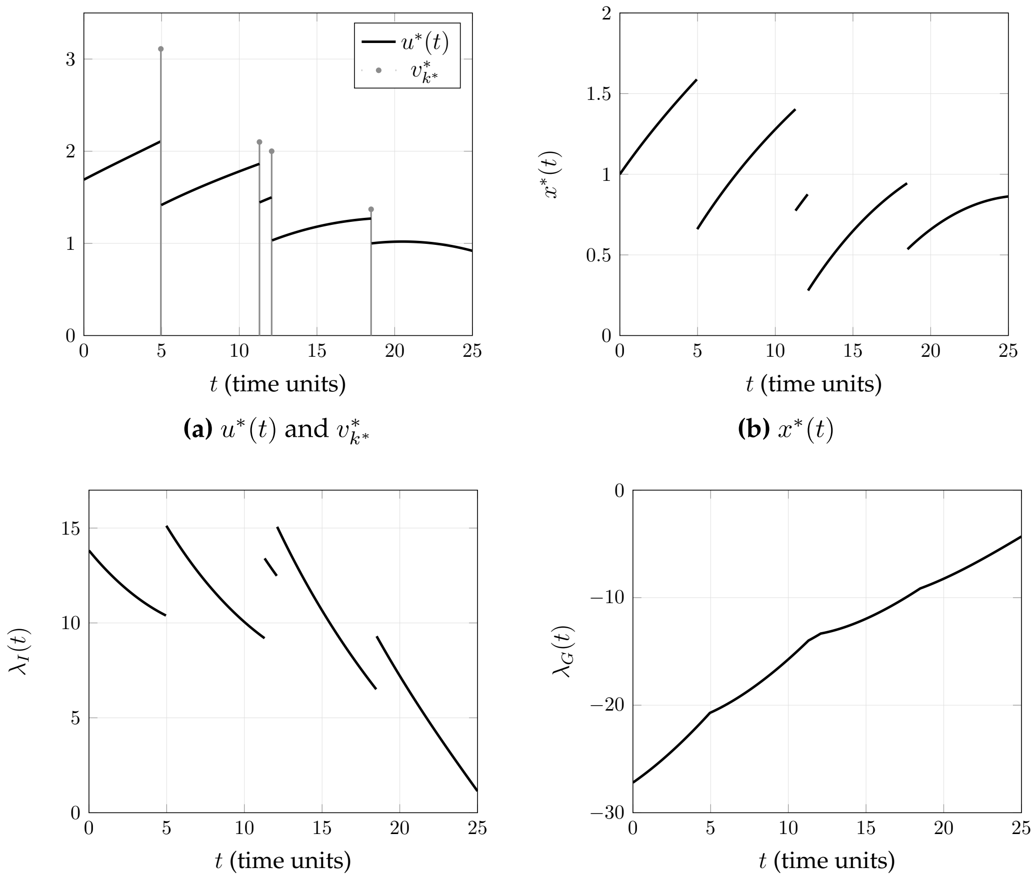


Figure 1.1 – Variation of the equilibrium profit of the government with the number of impulses

For $k^* = 4$, the open-loop Nash equilibrium strategies and the equilibrium state and co-state for the government and ITO are illustrated in Figure 1.2. It can be seen in Figure 1.2b that the resources of ITO continuously increase in time except at the times of strikes when some of the resources are destroyed. As shown in Figure 1.2a, the ITO invests effort in building its resources as the running payoff of ITO is increasing in the state. The co-state of ITO in Figure 1.2c jumps due to resource-dependent costs incurred at the time of the strike whereas the government's co-state is continuous in time (see Figure 1.2d) because government does not incur any state-dependent costs. The increase in government's co-state varies linearly with ITO's resources except at the time of the strike (see (1.10f)). This leads to a monotonic increase in government's co-state over time and therefore, at equilibrium, the government starts with a significant disruption of ITO's resources and the strikes that follow are less severe, and cause lower damage to the ITO's resources.



(a) $u^*(t)$ and $v^*_{k^*}$ **(b)** $x^*(t)$

(c) $\lambda_I(t)$ **(d)** $\lambda_G(t)$

Figure 1.2 – Equilibrium controls, and state and co-state trajectories.

1.7 Concluding remarks

In this paper, we studied a class of two-player nonzero-sum differential games where one player uses ordinary controls while the other uses impulse controls. We derived the necessary and sufficient conditions for the existence of an open-loop Nash equilibrium for this class of differential games. Then, specializing our results to a linear quadratic setting, we provide a non-linear optimization based approach for solving the open-loop Nash equilibrium strategies. We showed that the open-loop Nash equilibrium of a linear-state differential game with impulse control is unique and can have at most one impulse. Also, we derived expressions for the equilibrium timing and level of the impulse.

In this paper, we considered an open-loop information structure only. For future research, it would be interesting to determine the equilibrium under the closed-loop or feedback information structures. Another extension would be to allow both players to use both piecewise continuous and impulse controls.

1.8 Appendix

1.8.1 Proof of Theorem 1.1

Introduce the new state variable $\tilde{x}(t) : [0, T] \to \mathbb{R}$ with its dynamics given by

$$\begin{aligned}
&\dot{\tilde{x}}(t) = F_1(x(t), u(t)), \text{ for } t \in [0, T] \backslash \{\tau_1^*, \tau_2^*, ..., \tau_{k^*}^*\}, \\
&\tilde{x}(\tau_i^{*+}) - \tilde{x}(\tau_i^{*-}) = G_1(x(\tau_i^{*-}), v_i^*), \text{ for } i = \{1, 2, ..., k^*\}, \\
&\tilde{x}(0^-) = 0.
\end{aligned}$$

The objective function of Player 1 can now be expressed as $J_1(x_0, \tilde{u}, \tilde{v}^*) = \tilde{x}(T^+) + S_1(x(T^+))$. We define an augmented system as follows:

$$y(t) = \begin{bmatrix} x(t) \\ \tilde{x}(t) \end{bmatrix},$$

where $y(t) : [0, T] \to \mathbb{R}^{n+1}$. The dynamics of y at non-jump instants $\{\tau_1^*, \tau_2^*, ..., \tau_{k^*}^*\}$ are governed by

$$\dot{y}(t) = \begin{bmatrix} f(x(t), u(t)) \\ F_1(x(t), u(t)) \end{bmatrix} =: h(y(t), u(t)), \tag{1.29}$$

with the initial condition $y(0^-) = [x_0^T \; 0]^T$. An optimal trajectory x^* of (1.5) can be obtained from the optimal trajectory $y^*(t)$ of the augmented system by projection onto $\mathbb{R}^n$ parallel to the $\tilde{x}$ axis. The jump conditions on y can be represented as

$$y(\tau_i^{*+}) - y(\tau_i^{*-}) = \begin{bmatrix} g(x(\tau_i^{*-}), v_i^*) \\ G_1(x(\tau_i^{*-}), v_i^*) \end{bmatrix}, \text{ for } i = \{1, 2, ...k^*\}. \tag{1.30}$$

Now, we define a perturbed control $u_{w,I}(t)$, which is obtained by a needle variation in the optimal control, that is,

$$u_{w,I}(t) := \begin{cases} u^*(t) & \text{if } t \notin I \\ w & \text{if } t \in I \end{cases},$$

where $w \in \Omega_u$ and $I = (b - \epsilon a, b] \in [0, T]$, $\epsilon > 0$ is small, and $a > 0$ is arbitrary. We assume that $u^*(t)$ is continuous at b because we want $y^*(t)$ to be differentiable at $t = b$. Here, we also assume that $\tau_i^* \notin I, \ \forall i \in \{1, 2, ...k^*\}$. The linearized dynamics of the system in (1.29) around $y^*(t)$ is governed by

$$\dot{\varphi}(t) = \begin{bmatrix} f_x(x(t), u(t)) & 0_{n\times 1} \\ F_{1x}(x(t), u(t))^T & 0 \end{bmatrix} \varphi(t).$$

Suppose $I \subset (\tau_{i-1}^*, \tau_i^*)$. Let Φ_* denote the state transition matrix for the dynamics of φ such that, for $b \leq t \leq \tau_i^{*-}$, we have

$$y(t) = y^*(t) + \epsilon \Phi_*(t, b)\nu_w(b) + O(\epsilon), \tag{1.31}$$

where $\nu_w(b) = (h(y^*(b), w) - h(y^*(b), u^*(b)))\, a$ and $\frac{O(\epsilon)}{\epsilon} \to 0$ as $\epsilon \to 0$.
We introduce the matrix M_i to account for the change in the perturbed trajectory $y(t)$ due to the jump in $y(t)$,

$$M_i = \begin{bmatrix} I_{n\times n} + g_x(x(\tau_i^{*-}), v_i^*) & 0_{n\times 1} \\ (G_{1x}(x(\tau_i^{*-}), v_i^*))^T & 1 \end{bmatrix}.$$

Thus, we can write

$$y(\tau_i^{*+}) = y^*(\tau_i^{*+}) + \epsilon M_i \Phi_*(\tau_i^{*-}, b)\nu_w(b) + O(\epsilon).$$

Now, we can represent the terminal state as

$$y(T^+) = y^*(T^+) + \epsilon \delta(w, I) + O(\epsilon), \tag{1.32}$$

where $\delta(w, I) = \epsilon \Phi_*(T^+, \tau_{k^*}^{*+}) M_{k^*} \Phi_*(\tau_{k^*}^{*-}, \tau_{k^*-1}^{*+}) \cdots \ M_i \Phi_*(\tau_i^{*-}, b)\nu_w(b)$ is the infinitesimal change in the terminal state due to the needle variation in the control. The direction of $\delta(w, I)$ is dependent on w and b. By varying the parameters w and I, we can generate a cone $\vec{P}$ with vertex at $y^*(T^+)$ and rays $\rho(w, b)$ originating in the direction of $\delta(w, I)$. Here, $\vec{P}$ is not convex, so we consider various needle variations in the trajectory and concatenate them to obtain a terminal convex cone $TC(y^*(T^+))$. Therefore, there exists a non-zero vector $p \in \mathbb{R}^{n+1}$ such that

$$p^T(y(T^+) - y^*(T^+)) \geq 0, \tag{1.33}$$

where $y(T^+) - y^*(T^+) \in TC(y^*(T^+))$. Let us define p as

$$p = \begin{bmatrix} -S_{1x}(x(T^+)) \\ -1 \end{bmatrix},$$

which satisfies (1.33) because, otherwise, we can find a $y(T^+)$ such that $J(u) < J(u_{w,I})$. From (1.32) and (1.33), we get

$$p^T\Phi_*(T^+, \tau_{k^*}^{*+})M_{k^*}\Phi_*(\tau_{k^*}^{*-}, \tau_{k^*-1}^{*+})....M_i\Phi_*(\tau_i^{*-}, b)\nu_w(b) \geq 0. \tag{1.34}$$

Introduce an adjoint vector $q(t) : [0, t_N] \to \mathbb{R}^n$ with dynamics governed by

$$\dot{q}(t) = \begin{bmatrix} -(f_x(x^*(t), u^*(t))^T & -F_{1x}(x^*(t), u^*(t)) \\ 0 & 0 \end{bmatrix} q(t), \text{ for } t \in [0, T]\backslash\{\tau_1^*, \tau_2^*, ..., \tau_{k^*}^*\}, \tag{1.35}$$

$$q(T^+) = p, \tag{1.36}$$

$$q(\tau_i^{*-}) = \begin{bmatrix} (I + g_x(x^*(\tau_i^{*-}), v_i^*))^T & G_{1x}(x^*(\tau_i^{*-}), v_i^*) \\ 0 & 1 \end{bmatrix} q(\tau_i^{*+}), \text{ for } i = \{1, 2, ..., k^*\}. \tag{1.37}$$

Therefore, $\forall w \in \Omega_u, b \in [0, T]$, we can write (1.34) as

$$q(b)^T(h(y^*(b), w) - h(y^*(b), u^*(b))) \geq 0. \tag{1.38}$$

It is clear from (1.35) that the last component of q is a constant. From (1.36) and using the continuity of the last component of q in (1.37), we can set the last component of q to be equal to -1. So we can decompose $q(t)$ as

$$q(t) = \begin{bmatrix} -\lambda_1(t) \\ -1 \end{bmatrix}. \tag{1.39}$$

Substitute (1.39) in (1.35), (1.36), and (1.37) to obtain

$$\begin{aligned} \dot{\lambda_1}(t) =& - F_{1x}(x^*(t), u^*(t)) - (f_x(x^*(t), u^*(t))^T\lambda_1(t), \forall t \notin \{\tau_1^*, \tau_2^*, ..., \tau_{k^*}^*\}, \quad (1.40) \\ \lambda_1(T^+) =& S_{1x}(x(T^+)), \\ \lambda_1(\tau_i^{*-}) =& (I + (g_x(x^*(\tau_i^{*-}), v_i^*))^T)\lambda_1(\tau_i^{*+}) + G_{1x}(x^*(\tau_i^{*-}), v_i^*). \end{aligned}$$

Defining the Hamiltonian as

$$H_1(x(t), u(t), \lambda_1(t)) = F_1(x(t) + \lambda_1(t)^T f(x(t), u(t),$$

we can write the dynamics of $\lambda_1(t)$ in (1.40) as

$$\dot{\lambda_1}(t) = -H_{1x}(x^*(t), u^*(t), \lambda_1(t)).$$

From (1.38) and (1.39), we obtain

$$\begin{bmatrix} -\lambda_1(t)^T & -1 \end{bmatrix} \begin{bmatrix} f(x^*(t), w) - f(x^*(t), u^*(t)) \\ F_1(x^*(t), w) - F_1(x^*(t), u^*(t)) \end{bmatrix} \geq 0.$$

Using the definition of the Hamiltonian, we obtain

$$H_1(x^*(t), u^*(t), \lambda_1(t)) \geq H_1(x^*(t), w, \lambda_1(t)), \text{ for } t \in [0, T] \backslash \{\tau_1^*, \tau_2^*, ..., \tau_{k^*}^*\}.$$

1.8.2 Proof of Theorem 1.2

From (1.4a), $u^*(.)$ is Player 1's best response to Player 2's equilibrium strategy $\tilde{v}^*$. Following Theorem 1.1, conditions (1.10a), (1.10b), (1.10d), (1.10e), (1.10h), and (1.10i) provide the necessary conditions for $u^*(.)$ to be the best response for $\tilde{v}^*$.

Next, following (1.4b), Player 2's best reponse $\tilde{v}^*$ to Player 1's equilibrium strategy $u^*(.)$ is an impulse optimal control problem (1.7). The necessary conditions for optimality associated with an impulse optimal control problem were studied in Blaquière (1977b, Theorem 1.2) and Chahim et al. (2012). We list these conditions for the impulse optimal control problem (1.7) below.

The equilibrium level of impulse at each time instant τ_i^* ($i = 1, 2, \cdots, k$) is given by

$$v_i^* = \arg\max_{v_i \in \Omega_v} H_2^I(x^*(\tau_i^{*-}), v_i, \lambda_2(\tau_i^{*+})).$$

The maximized impulse Hamiltonian at the impulse instant τ_i ($i = 1, 2, \cdots, k$) is given by

$$H_2^{I^*}(x^*(\tau_i^{*-}), \lambda_2(\tau_i^{*+})) = H_2^I(x^*(\tau_i^{*-}), v_i^*, \lambda_2(\tau_i^{*+})).$$

During the non-impulse instants $t \in [0, T] \backslash \{\tau_1^*, \tau_2^*, \cdots, \tau_k^*\}$, the state and co-state equations satisfy

$$\begin{aligned} &\dot{x}^*(t) = f(x^*(t), u^*(t)), \ x^*(0^-) = x_0, \\ &\dot{\lambda}_2(t) = -H_{2x}^*(x^*(t), u^*(t), \lambda_2(t)), \ \lambda_2(T^+) = S_{2x}(x^*(T^+)). \end{aligned}$$

At the impulse instants τ_i ($i = 1, 2, \cdots, k$), the state and co-state variables admit jumps according to

$$x^*(\tau_i^{*+}) = x^*(\tau_i^{*-}) + g(x^*(\tau_i^{*-}), v_i^*),$$

$$\lambda_2(\tau_i^{*-}) = \lambda_2(\tau_i^{*+}) + H_{2x}^{I^*}(x^*(\tau_i^{*-}), \lambda_2(\tau_i^{*+})),$$

and the Hamiltonian function satisfies

$$H_2(x^*(\tau_i^{*+}), u^*(\tau_i^{*+}), \lambda_2(\tau_i^{*+})) - H_2(x^*(\tau_i^{*-}), u^*(\tau_i^{*-}), \lambda_2(\tau_i^{*-})) \begin{cases} > 0 & \text{for } \tau_i^* = 0 \\ = 0 & \text{for } \tau_i^* \in (0, T) \\ < 0 & \text{for } \tau_i^* = T \end{cases}.$$

The above listed conditions constitute the necessary conditions (1.10c), (1.10d), (1.10f), (1.10g), (1.10h), (1.10j), and (1.10k).

1.8.3 Proof of Theorem 1.3

Suppose a pair of controls $(u^*(.), \tilde{v}^*)$ satisfy the conditions in Theorem 1.2 and $x^*(t)$ is the corresponding state trajectory. We have to show that $u^*(.)$ and $\tilde{v}^*$ are mutual best response strategies. Firstly, for Player 1, we define the difference for any $u(.) \in \mathcal{U}$,

$$\begin{aligned} \Delta J_1 &= J_1(u^*(.), \tilde{v}^*) - J_1(u(.), \tilde{v}^*) \\ &= \int_0^T F_1(x^*(t), u^*(t))dt + S_1(x^*(T)) + \sum_{i=1}^{k^*} G_1(x^*(\tau_i^{*-}), v_i^*) \\ &- \int_0^T F_1(x(t), u(t))dt - S_1(x(T)) - \sum_{i=1}^{k^*} G_1(x(\tau_i^{*-}), v_i^*), \end{aligned} \tag{1.41}$$

We use the definition of the Hamiltonian of Player 1 in (1.6a) to obtain

$$\begin{aligned} \Delta J_1 = &\int_0^T H_1(x^*(t), u^*(t), \lambda_1(t))dt + S_1(x^*(T)) - \int_0^T H_1(x(t), u(t), \lambda_1(t))dt \\ &+ \int_0^T \lambda_1(t)^T(\dot{x}(t) - \dot{x}^*(t))dt - S_1(x(T)) \\ &+ \sum_{i=1}^{k^*} G_1(x^*(\tau_i^{*-}), v_i^*) - G_1(x(\tau_i^{*-}), v_i^*). \end{aligned}$$

Recall that for any $x(t)$, we have

$$H_1^*(x(t), \lambda_1(t)) = \max_{u(t) \in \Omega_u} H_1(x(t), u(t), \lambda_1(t)).$$

It follows from the above equation that

$$H_1^*(x(t), \lambda_1(t)) \geq H_1(x(t), u(t), \lambda_1(t)), \ \forall u(t) \in \Omega_u.$$

Using (1.10a) and the above inequality, we obtain

$$H_1(x^*(t), u^*(t), \lambda_1(t)) - H_1(x(t), u(t), \lambda_1(t)) \geq H_1^*(x^*(t), \lambda_1(t)) - H_1^*(x(t), \lambda_1(t)).$$

From the concavity of the maximized Hamiltonian $H_1^*(x(t), \lambda_1(t))$ in $x(t)$,

$$H_1^*(x(t), \lambda_1(t)) - H_1^*(x^*(t), \lambda_1(t)) \leq H_{1x}^*(x^*(t), \lambda_1(t))^T(x(t) - x^*(t)).$$

Using (1.10e), we get for all $t \neq \{\tau_1^*, \cdots, \tau_k^*\}$,

$$\begin{aligned} H_1^*(x(t), \lambda_1(t)) - H_1^*(x^*(t), \lambda_1(t)) \leq & H_{1x}^*(x^*(t), \lambda_1(t))^T(x(t) - x^*(t)) \\ & - \dot{\lambda}_1(t)^T(x(t) - x^*(t)). \end{aligned}$$

Similarly, from the concavity of $S_1(x(T))$ in $x(T)$, and $\lambda_1(T) = S_{1x}(x^*(T))$, we obtain

$$S_1(x(T)) - S_1(x^*(T)) \leq S_{1x}(x^*(T))^T(x(T) - x^*(T)) = \lambda_1(T)^T(x(T) - x^*(T)).$$

Using the above inequalities, we obtain

$$\begin{aligned} \Delta J_1 \geq & \sum_{i=0}^{k^*} \Big(\int_{\tau_i^{*+}}^{\tau_{i+1}^{*-}} \dot{\lambda}_1(t)^T(x(t) - x^*(t))dt + \int_{\tau_i^{*+}}^{\tau_{i+1}^{*-}} \lambda_1(t)^T(\dot{x}(t) - \dot{x}^*(t))dt \Big) \\ & + \sum_{i=1}^{k^*} \big(G_1(x^*(\tau_i^{*-}), v_i^*) - G_1(x(\tau_i^{*-}), v_i^*) \big) + \lambda_1(T)^T(x^*(T) - x(T)), \end{aligned} \quad (1.42)$$

where we define $\tau_0^* := 0$ and $\tau_{k+1}^* := T$. Recall that we have made the assumption that there can only be interior impulse instants. τ_0^* and τ_{k+1}^* are used to simplify the notation, and are not the impulse instants.

From

$$\int_{t_1^+}^{t_2^-} \frac{d}{dt}(\lambda^T(t)x(t))dt = \int_{t_1^+}^{t_2^-} [\lambda(t)^T \dot{x}(t) + \dot{\lambda}(t)^T x(t)]dt, \quad (1.43)$$

and (1.42), we obtain

$$\Delta J_1 \geq \sum_{i=0}^{k^*} \left(\int_{\tau_i^{*+}}^{\tau_{i+1}^{*-}} \frac{d}{dt} \big(\lambda_1(t)^T(x(t) - x^*(t))\big) \, dt \right) + \lambda_1(T)^T(x^*(T) - x(T))$$

$$
\begin{aligned}
&+\sum_{i=1}^{k^*} G_1(x^*(\tau_i^{*-}),v_i^*) - G_1(x(\tau_i^{*-}),v_i^*),\\
&= \lambda_1(\tau_1^{*-})^T(x(\tau_1^{*-}) - x^*(\tau_1^{*-})) + \lambda_1(\tau_2^{*-})^T(x(\tau_2^{*-}) - x^*(\tau_2^{*-}))\\
&\quad - \lambda_1(\tau_1^{*+})^T(x(\tau_1^{*+}) - x^*(\tau_1^{*+})) + \cdots + \lambda_1(T)^T(x(T) - x^*(T))\\
&\quad - \lambda_1(\tau_k^{*+})^T(x(\tau_k^{*+}) - x^*(\tau_k^{*+})) + \lambda_1(T)^T(x^*(T) - x(T))\\
&\quad + \sum_{i=1}^{k^*} G_1(x^*(\tau_i^{*-}),v_i^*) - G_1(x(\tau_i^{*-}),v_i^*),
\end{aligned}
$$

where we have used the fact that there is discontinuity in $\lambda_1(t)^T x(t)$ at the jump instants. Also, $x^*(\tau_0^{*-}) = x(\tau_0^{*-}) = x_0$ since we have assumed that there is no impulse at $t = 0$. From (1.10h), we can write $x(\tau_i^{*+}) - x^*(\tau_i^{*+}) = x(\tau_i^{*-}) + g(x(\tau_i^{*-}),v_i^*) - x^*(\tau_i^{*-}) - g(x^*(\tau_i^{*-}),v_i^*)$. Rearranging the terms on the right-hand side of the above expression and using (1.10i), we get

$$
\begin{aligned}
\Delta J_1 \geq &\sum_{i=1}^{k^*} \big(G_1(x^*(\tau_i^{*-}),v_i^*) + \lambda_1(\tau_i^{*+})^T g(x^*(\tau_i^{*-}),v_i^*) - G_1(x(\tau_i^{*-}),v_i^*)\\
&- \lambda_1(\tau_i^{*+})^T g(x(\tau_i^{*-}),v_i^*)\big) + \sum_{i=1}^{k^*} \big(G_{1x}(x^*(\tau_i^{*-}),v_i^*)^T(x(\tau_i^{*-}) - x^*(\tau_i^{*-}))\\
&+ \lambda_1(\tau_i^{*+})^T g_x(x^*(\tau_i^{*-}),v_i^*)(x(\tau_i^{*-}) - x^*(\tau_i^{*-}))\big).
\end{aligned}
$$

From the concavity of $G_1(x(t),v) + \lambda_1^T(t) g(x(t),v)$ in $x(t)$,

$$
\begin{aligned}
&G_1(x^*(\tau_i^{*-}),v_i^*) + \lambda_1(\tau_i^{*+})^T g(x^*(\tau_i^{*-}),v_i^*) - G_1(x(\tau_i^{*-}),v_i^*) - \lambda_1(\tau_i^{*+})^T g(x(\tau_i^{*-}),v_i^*)]\\
&\geq [G_{1x}(x^*(\tau_i^{*-}),v_i^*)^T + \lambda_1(\tau_i^{*+})^T g_x(x^*(\tau_i^{*-}),v_i^*)](x^*(\tau_i^{*-}) - x(\tau_i^{*-})).
\end{aligned}
$$

It follows from the above relation that $\Delta J_1 \geq 0$, and this implies that $u^*(.)$ is the best response to $\tilde{v}^*$. For Player 2's impulse optimal control problem (1.7), we note that Player 2 uses impulse controls (and not the piecewise continuous controls), and the Hamiltonian function $H_2(x(t),u^*(t),\lambda_2(t))$ is concave in $x(t)$ for each t, and the impulse Hamiltonian $H_2^I(x(t),v,\lambda_2(t))$ is concave in $(x(t),v)$ for each $\lambda_2(t)$. Then from Seierstad (1981, Theorem 1) and Seierstad and Sydsæter (1987, Theorem 8, pages 198-199), it follows that the necessary conditions (1.10d), (1.10f)-(1.10k) are also sufficient for Player 2's impulse optimal control problem (1.4b), that is, $\tilde{v}^*$ is a best response to $u^*(.)$.

1.8.4 Proof of Theorem 1.4

From (1.6a), we have

$$\begin{aligned} H_1(x(t), u(t), \lambda_1(t)) =& \frac{1}{2}(x(t)^T W_1 x(t) + 2w_1^T x(t) + u(t)^T R_1 u(t) + 2d_1^T u(t)) \\ &+ \lambda_1(t)^T (Ax(t) + Bu(t)). \end{aligned}$$

From (1.10a), and from the negative definiteness of R_1, we get

$$\begin{aligned} &H_{1u}(x(t), u(t), \lambda_1(t))|_{u^*(t)} = 0 \Rightarrow R_1 u(t) + d_1 + B^T \lambda_1(t) = 0 \\ &\Rightarrow u^*(t) = -R_1^{-1}(B^T \lambda_1(t) + d_1). \end{aligned} \tag{1.44}$$

The maximized Hamiltonian of Player 1 is given by

$$\begin{aligned} H_1^*(x(t), \lambda_1(t)) =& \frac{1}{2}(x(t)^T W_1 x(t) - (B^T \lambda_1(t) + d_1)^T R_1^{-1}(B^T \lambda_1(t) + d_1)) \\ &+ (w_1 + A^T \lambda_1(t))^T x(t). \end{aligned} \tag{1.45}$$

From (1.8), we get

$$\begin{aligned} H_2(x(t), u^*(t), \lambda_2(t)) =& \frac{1}{2}\big(x(t)^T W_2 x(t) + 2w_2^T x(t) + u^*(t)^T R_2 u^*(t) + 2d_2^T u^*(t)\big) \\ &+ \lambda_2(t)^T (Ax(t) + Bu^*(t)). \end{aligned}$$

We substitute (1.44) in the above expression to obtain

$$\begin{aligned} &H_2(x(t), \lambda_1(t), \lambda_2(t)) \\ &= \frac{1}{2} x(t)^T W_2 x(t) + (w_2 + A^T \lambda_2(t))^T x(t) \\ &\quad + \frac{1}{2}(R_2 R_1^{-1} B^T \lambda_1(t) - 2B^T \lambda_2(t) + R_2 R_1^{-1} d_1 - 2d_2)^T R_1^{-1}(B^T \lambda_1(t) + d_1). \end{aligned} \tag{1.46}$$

From (1.9),

$$\begin{aligned} H_2^I(x(\tau_i^-), v_i, \lambda_2(\tau_i^+)) =& \frac{1}{2}\big(x(\tau_i^-)^T Z_2 x(\tau_i^-) + 2q_2^T x(\tau_i^-) + v_i^T P_2 v_i + 2p_2^T v_i\big) \\ &+ \lambda_2(\tau^+)^T Q v_i. \end{aligned} \tag{1.47}$$

From (1.10g), (1.47), and from the negative definiteness of P_2, we get

$$\begin{aligned} &H_{2v_i}^I(x(\tau_i^-), v_i, \lambda_2(\tau_i^+))|_{v_i^*} = 0 \Rightarrow P_2 v_i + p_2 + Q^T \lambda_2(\tau_i^+) = 0 \\ &\Rightarrow v_i^* = -P_2^{-1}(Q^T \lambda_2(\tau_i^+) + p_2). \end{aligned} \tag{1.48}$$

Substituting v_i^* in (1.47), we get

$$H_2^{I*}(x(\tau_i^-), \lambda_2(\tau_i^+)) = \frac{1}{2}\left(x(\tau_i^-)^T Z_2 x(\tau_i^-) - (Q^T\lambda_2(\tau_i^+) + 2p_2)^T P_2^{-1}(Q^T\lambda_2(\tau^+)\right) + q_2^T x(\tau_i^-). \quad (1.49)$$

From (1.10d)-(1.10f) and (1.44)-(1.46), the optimal state and co-state trajectories for non-impulse instants (1.16b) are obtained as

$$\begin{aligned}\dot{x}^*(t) =& Ax^*(t) - BR_1^{-1}(B^T\lambda_1(t) + d_1),\ x(0) = x_0,\\ \dot{\lambda}_1(t) =& -A^T\lambda_1(t) - W_1 x^*(t) - w_1, \lambda_1(T^+) = S_1 x(T^+) + s_1,\\ \dot{\lambda}_2(t) =& -A^T\lambda_2(t) - w_2 - W_2 x^*(t), \lambda_2(T^+) = S_2 x(T^+) + s_2.\end{aligned}$$

From (1.10h)-(1.10j) and (1.48)-(1.49), the jump conditions at the impulse instants (1.16e) are obtained as

$$\begin{aligned}x^*(\tau_i^{*+}) =& x^*(\tau_i^{*-}) - QP_2^{-1}(Q^T\lambda_2(\tau_i^{*+}) + p_2),\\ \lambda_1(\tau_i^{*-}) =& \lambda_1(\tau_i^{*+}) + Z_1 x^*(\tau_i^{*-}) + q_1,\\ \lambda_2(\tau_i^{*-}) =& \lambda_2(\tau_i^{*+}) + Z_2 x^*(\tau_i^{*-}) + q_2.\end{aligned}$$

Finally, from (1.10k) and (1.46) and rearranging terms, we get (1.16f).

1.8.5 Proof of Theorem 1.5

From (1.16b), $y(t)$ for $t \in (\tau_i^*, \tau_{i+1}^*)$ is evaluated as

$$y(t) = e^{M(t-\tau_i^{*+})} y(\tau_i^{*+}) + e^{Mt}\int_{\tau_i^{*+}}^{t} e^{-Ms} C\, ds.$$

Next, using (1.17), (1.18), and (1.16e), we obtain (1.20a). Since we assumed interior impulse instants, that is, $\tau_i^* \in (0, T)$ for $i \in \{1, 2, \cdots, k^*\}$, the Hamiltonian continuity condition (1.16f) holds true with equality. Rewriting (1.16f) using (1.19), we get

$$\begin{aligned}H_2(y(\tau_i^{*+})) - H_2(y(\tau_i^{*-})) =& \frac{1}{2} y(\tau_i^{*+})^T \mathbf{A_1} y(\tau_i^{*+}) + \mathbf{b}^T y(\tau_i^{*+}) + \mathbf{c}\\ &- \left(\frac{1}{2} y(\tau_i^{*-})^T \mathbf{A_1} y(\tau_i^{*-}) + \mathbf{b}^T y(\tau_i^{*-}) + \mathbf{c}\right).\end{aligned}$$

Using (1.16e) in the above equation, we obtain (1.20b). Finally, (1.20c) and (1.20d), which are expressed in open-loop form, are obtained from (1.16a) and (1.16d).

1.8.6 Proof of Theorem 1.7

The necessary conditions (1.16) from Theorem 1.4 are given by
For $t \notin \{\tau_1^*, \tau_2^*, ...\tau_{k^*}^*\}$,

$$\dot{x}^*(t) = Ax^*(t) - \frac{B^2}{R_1}\lambda_1(t),\ x(0^-) = x_0,$$
$$\dot{\lambda}_1(t) = -A\lambda_1(t) - w_1,\ \lambda_1(T^+) = s_1,$$
$$\dot{\lambda}_2(t) = -A\lambda_2(t) - w_2,\ \lambda_2(T^+) = s_2.$$

For $t \in \{\tau_1^*, \tau_2^*, ...\tau_{k^*}^*\}$,

$$x^*(\tau_i^{*+}) = x^*(\tau_i^{*-}) - \frac{Q^2}{P_2}\lambda_2(\tau_i^{*+}), \tag{1.50a}$$
$$\lambda_1(\tau_i^{*-}) = \lambda_1(\tau_i^{*+}) + q_1, \tag{1.50b}$$
$$\lambda_2(\tau_i^{*-}) = \lambda_2(\tau_i^{*+}). \tag{1.50c}$$

The Hamiltonian continuity condition (1.16f) is given by

$$w_2(x^*(\tau_i^{*+}) - x^*(\tau_i^{*-})) + A(\lambda_2(\tau_i^{*+})x(\tau_i^{*+}) - \lambda_2(\tau_i^*)x(\tau_i^*))$$
$$- \frac{B^2}{R_1}(\lambda_1(\tau_i^{*+})\lambda_2(\tau_i^{*+}) - \lambda_1(\tau_i^{*-})\lambda_2(\tau_i^{*-})) = 0.$$

Substituting (1.50a), (1.50b), and (1.50c) in the above equation, we obtain

$$-\left((w_2 + A\lambda_2(\tau_i^{*-}))\frac{Q^2}{P_2} - \frac{B^2}{R_1}q_1\right)\lambda_2(\tau_i^{*-}) = 0.$$

The above quadratic equation has two solutions: $\lambda_2(\tau_i^*) = 0$ and $\lambda_2(\tau_i^*) = \frac{\frac{-w_2Q^2}{P_2} + \frac{B^2q_1}{R_1}}{A\frac{Q^2}{P_2}}$ for $A \neq 0$. From (1.16d), with $\lambda_2(\tau_i^{*-}) = 0$, we get $v_i^* = 0$ for all $i \in \{1, 2, \cdots, k^*\}$. Therefore, from Assumption 1.(e), we have that τ_i^* is not an impulse instant. This implies that

$$\lambda_2(\tau_i^{*-}) = \frac{\frac{-w_2Q^2}{P_2} + \frac{B^2q_1}{R_1}}{A\frac{Q^2}{P_2}},\ i \in \{1, 2, \cdots, k^*\}, \tag{1.51}$$

is the valid solution for $\frac{Q^2w_2}{P_2} \neq \frac{B^2q_1}{R_1}$ for which $v_i^* \neq 0$. From (1.50c), we have that there are no jumps in the co-state of Player 2, and this implies that, for $A \neq 0$, $\lambda_2(t)$ is computed as

$$\lambda_2(t) = -\frac{w_2}{A} + (s_2 + \frac{w_2}{A})e^{A(T-t)},\ t \in [0,\ T]. \tag{1.52}$$

Clearly, $\lambda_2(t)$ is strictly monotone in t, and can take the value $\frac{\frac{-w_2 Q^2}{P_2}+\frac{B^2 q_1}{R_1}}{A\frac{q^2}{p_2}}$ at most once, that is, $k^* \leq 1$. Using (1.52) and (1.51), we obtain

$$\tau^* = T + \frac{1}{A}\ln\left(\left(\frac{Q}{B}\right)^2 \frac{R_1}{P_2}\left(\frac{As_2}{w_2}+1\right)\frac{w_2}{q_1}\right).$$

For $\tau^* > 0$, the parameters should satisfy $T + \frac{1}{A}\ln\left(\left(\frac{Q}{B}\right)^2 \frac{R_1}{P_2}\left(\frac{As_2}{w_2}+1\right)\frac{w_2}{q_1}\right) > 0$, while for $\tau^* < T$, the inequality,

$$\frac{1}{A}\ln\left(\left(\frac{Q}{B}\right)^2 \frac{R_1}{P_2}\left(\frac{As_2}{w_2}+1\right)\frac{w_2}{q_1}\right) < 0$$

should hold. The impulse level is calculated from (1.16d) and (1.51) to obtain (1.28).

References

Azimzadeh, P. (2019). A zero-sum stochastic differential game with impulses, precommitment, and unrestricted cost functions. *Applied Mathematics & Optimization*, 79(2):483–514.

Aïd, R., Basei, M., Callegaro, G., Campi, L., and Vargiolu, T. (2020). Nonzero-sum stochastic differential games with impulse controls: A verification theorem with applications. *Mathematics of Operations Research*, 45(1):205–232.

Başar, T. and Olsder, G. J. (1999). *Dynamic Noncooperative Game Theory, 2nd edition*. Society for Industrial and Applied Mathematics, Philadelphia, PA.

Barclay, A., Gill, P. E., and Ben Rosen, J. (1998). Sqp methods and their application to numerical optimal control. In Schmidt, W. H., Heier, K., Bittner, L., and Bulirsch, R., editors, *Variational Calculus, Optimal Control and Applications*, pages 207–222, Basel. Birkhäuser Basel.

Başar, T., Haurie, A., and Zaccour, G. (2018). Nonzero-Sum Differential Games. In Başar, T. and Zaccour, G., editors, *Handbook of Dynamic Game Theory*, pages 61–110. Springer International Publishing, Cham.

Basei, M. (2019). Optimal price management in retail energy markets: an impulse control problem with asymptotic estimates. *Mathematical Methods of Operations Research*, 89(3):355–383.

Basei, M., Cao, H., and Guo, X. (2019). Nonzero-sum stochastic games and mean-field games with impulse controls.

Bensoussan, A. and Lions, J.-L. (1982). *Contrôle Impulsionnel et Inéquations Quasi-Variationelles*. Bordas, Paris, France.

Bensoussan, A. and Lions, J.-L. (1984). *Impulse Control and Quasi-Variational Inequalities*. Gauthier-Villars, Paris, France.

Bensoussan, A., Long, H., Perera, S., and Sethi, S. (2012). Impulse control with random reaction periods: A central bank intervention problem. *Operations Research Letters*, 40(6):425 – 430.

Bernhard, P., El Farouq, N., and Thiery, S. (2006). An impulsive differential game arising in finance with interesting singularities. In Haurie, A., Muto, S., Petrosjan, L. A., and Raghavan, T. E. S., editors, *Advances in Dynamic Games: Applications to Economics, Management Science, Engineering, and Environmental Management*, pages 335–363. Birkhäuser, Boston, MA.

Bertola, G., Runggaldier, W. J., and Yasuda, K. (2016). On classical and restricted impulse stochastic control for the exchange rate. *Applied Mathematics & Optimization*, 74(2):423–454.

Blaquière, A. (1977a). Differential games with piece-wise continuous trajectories. In Hagedorn, P., Knobloch, H. W., and Olsder, G. J., editors, *Differential Games and Applications*, pages 34–69. Springer, Berlin.

Blaquière, A. (1977b). Necessary and sufficient conditions for optimal strategies in impulsive control and application. In Aoki, M. and Morzzolla, A., editors, *New Trends in Dynamic System Theory and Economics*, pages 183–213. Academic Press, New York.

Blaquière, A. (1979). Necessary and sufficient conditions for optimal strategies in impulsive control. In Lui, P. and Roxin, E., editors, *Differential Games and Control Theory III, Part A*, pages 1 – 28. Marcel Dekker, New York.

Blaquière, A. (1985). Impulsive optimal control with finite or infinite time horizon. *Journal of Optimization Theory and Applications*, 46(4):431–439.

Byrd, R. H., Hribar, M. E., and Nocedal, J. (1999). An interior point algorithm for large-scale nonlinear programming. *SIAM Journal on Optimization*, 9(4):877 – 900.

Büskens, C. and Maurer, H. (2000). Sqp-methods for solving optimal control problems with control and state constraints: adjoint variables, sensitivity analysis and real-time control. *Journal of Computational and Applied Mathematics*, 120(1):85 – 108.

Campi, L. and De Santis, D. (2020). Nonzero-sum stochastic differential games between an impulse controller and a stopper. *Journal of Optimization Theory and Applications*, 186(2):688–724.

Chahim, M. (2013). *Impulse control maximum principle: Theory and applications*. PhD thesis, Tilburg University.

Chahim, M., Brekelmans, R., den Hertog, D., and Kort, P. (2013). An impulse control approach to dike height optimization. *Optimization Methods and Software*, 28(3):458–477.

Chahim, M., Grass, D., Hartl, R. F., and Kort, P. M. (2017). Product innovation with lumpy investment. *Central European Journal of Operations Research*, 25(1):159–182.

Chahim, M., Hartl, R. F., and Kort, P. M. (2012). A tutorial on the deterministic Impulse Control Maximum Principle: Necessary and sufficient optimality conditions. *European Journal of Operational Research*, 219(1):18–26.

Chang, D., Wang, H., and Wu, Z. (2013). Maximum principle for non-zero sum differential games of BSDEs involving impulse controls. In *Proceedings of the 32nd Chinese Control Conference*, pages 1564–1569.

Chang, D. and Wu, Z. (2015). Stochastic maximum principle for non-zero sum differential games of FBSDEs with impulse controls and its application to finance. *Journal of Industrial & Management Optimization*, 11(1):27–40.

Chikrii, A. A. and Matichin, I. (2005). Linear differential games with impulse control of players. *Proceedings of the Steklov Institute of Mathematics*, Suppl. 1:S68–S81.

Chikrii, A. A., Matychyn, I. I., and Chikrii, K. A. (2007). Differential games with impulse control. In Jørgensen, S., Quincampoix, M., and Vincent, T. L., editors, *Advances in Dynamic Game Theory. Annals of the International Society of Dynamic Games*, volume 9, pages 37–55. Birkhäuser, Boston.

Cosso, A. (2013). Stochastic Differential Games Involving Impulse Controls and Double-Obstacle Quasi-variational Inequalities. *SIAM Journal on Control and Optimization*, 51(3):2102–2131.

Crettez, B. and Hayek, N. (2014). Terrorists' eradication versus perpetual terror war. *Journal of Optimization Theory and Applications*, 160:679–702.

Eastham, J. F. and Hastings, K. J. (1988). Optimal impulse control of portfolios. *Mathematics of Operations Research*, 13:588—-605.

El Asri, B. (2013). Deterministic minimax impulse control in finite horizon: the viscosity solution approach. *ESAIM: Control, Optimisation and Calculus of Variations*, 19(1):63–77.

El Asri, B. and Mazid, S. (2018). Zero-sum stochastic differential game in finite horizon involving impulse controls. *Applied Mathematics & Optimization*. ePub ahead of print, September 24.

El Farouq, N., Barles, G., and Bernhard, P. (2010). Deterministic minimax impulse control. *Applied Mathematics and Optimization*, 61(3):353 – 378.

Elliott, J., Nigel, and Kalton, J. (1972). The existence of value in differential games. *Memoirs of the American Mathematical Society*, page 126.

Engwerda, J. C. (2005). *LQ Dynamic Optimization and Differential Games*. Wiley, New York.

Erdlenbruch, K., Jean-Marie, A., Moreaux, M., and Tidball, M. (2013). Optimality of impulse harvesting policies. *Economic Theory*, 52(2):429 – 459.

Etesami, S. R. and Başar, T. (2019). Dynamic games in cyber-physical security: An overview. *Dynamic Games and Applications*, 9(4):884–913.

Ferrari, G. and Koch, T. (2019). On a strategic model of pollution control. *Annals of Operations Research*, 275(2):297–319.

Geering, H. P. (1976). Continuous-Time Optimal Control Theory for Cost Functionals Including Discrete State Penalty Terms. *IEEE Transactions on Automatic Control*, 21(6):866–869.

Getz, W. M. and Martin, D. H. (1980). Optimal control systems with state variable jump discontinuities. *Journal of Optimization Theory and Applications*, 31(2):195 – 205.

Grames, J., Grass, D., Kort, P. M., and Prskawetz, A. (2019). Optimal investment and location decisions of a firm in a flood risk area using impulse control theory. *Central European Journal of Operations Research*, 27(4):1051–1077.

Grass, D. and Chahim, M. (2012). Numerical algorithm for impulse control models. Working paper.

Haurie, A., Krawczyk, J. B., and Zaccour, G. (2012). *Games and Dynamic Games*. World Scientific, Singapore.

Hou, S. H. and Wong, K. H. (2011). Optimal Impulsive Control Problem with Application to Human Immunodeficiency Virus Treatment. *Journal of Optimization Theory and Applications*, 151(2):385–401.

Kort, P. M. (1989). *Optimal Dynamic Investment Policies of a Value Maximizing Firm*. Springer, Berlin.

Leander, R., Lenhart, S., and Protopopescu, V. (2015). Optimal control of continuous systems with impulse controls. *Optimal Control Applications and Methods*, 36(4):535–549.

Liu, Y., Teo, K. L., Jennings, L. S., and Wang, S. (1998). On a Class of Optimal Control Problems with State Jumps. *Journal of Optimization Theory and Applications*, 98(1):65 – 82.

Neuman, C. E. and Costanza, V. (1990). Deterministic Impulse Control in Native Forest Ecosystems Management. *Journal of Optimization Theory and Applications*, 66(2):173 – 196.

Novak, A., Feichtinger, G., and Leitmann, G. (2010). A Differential Game Related to Terrorism: Nash and Stackelberg Strategies. *Journal of Optimization Theory and Applications*, 144:533–555.

Perera, S., Gupta, V., and Buckley, W. (2020). Management of online server congestion using optimal demand throttling. *European Journal of Operational Research*, 285(1):324 – 342.

Powell, M. J. D. (1978). A fast algorithm for nonlinearly constrained optimization calculations. In Watson, G. A., editor, *Numerical Analysis*, pages 144–157, Berlin, Heidelberg. Springer Berlin Heidelberg.

Reddy, P. V., Wrzaczek, S., and Zaccour, G. (2016). Quality effects in different advertising models - An impulse control approach. *European Journal of Operational Research*, 255(3):984–995.

Seierstad, A. (1981). *Necessary Conditions and Sufficient Conditions for Optimal Control with Jumps in the State Variables*. Memorandum from Institute of Economics. University of Oslo, Oslo.

Seierstad, A. and Sydsæter, K. (1987). *Optimal Control Theory with Economic Applications*. Advanced textbooks in economics. North-Holland, Amsterdam.

Sussmann, H. J. (1999). A maximum principle for hybrid optimal control problems. In *Proceedings of the 38th IEEE Conference on Decision and Control*, volume 1, pages 425–430, Phoenix, AZ, USA.

Taynitskiy, V., Gubar, E., and Zhu, Q. (2019). Optimal Impulse Control of SIR Epidemics Over Scale-Free Networks. In Song, J. B., Li, H., and Coupechoux, M., editors, *Game Theory for Networking Applications*, pages 119–129. Springer International Publishing, Cham.

Waltz, R., Morales, J., and Nocedal, J. (2006). An interior algorithm for nonlinear optimization that combines line search and trust region steps. *Math. Program.*, 107:391–408.

Wu, C. and Teo, K. L. (2006). Global impulsive optimal control computation. *Journal of Industrial and Management Optimization*, 2(4):435–450.

Yong, J. (1994). Zero-sum differential games involving impulse controls. *Applied Mathematics and Optimization*, 29(3):243–261.

Zhang, F. (2011). Stochastic differential games involving impulse controls. *ESAIM: Control, Optimisation and Calculus of Variations*, 17(3):749–760.

Chapter 2

Open-loop and feedback Nash equilibria in scalar linear-state differential games with impulse control

Abstract

We consider a deterministic two-player linear-state differential game, where Player 1 uses piecewise continuous controls, while Player 2 implements impulse controls. When the impulse instants are not the decision variables for Player 2, but provided exogenously, we recover the classical result that both open-loop and feedback Nash equilibria coincide for this class of games. When the number and timing of impulse instants are decision variables of Player 2, we show that the classical result no longer holds, that is, open-loop and feedback Nash equilibria are different. We show that the impulse level is constant in both equilibria. More importantly, in the open-loop case, we show that the equilibrium number of impulses is at most three, while there can be at most two impulses in the feedback case.

2.1 Introduction

Differential games are used to study competitive strategic interactions between multiple agents (players) over time (see Başar and Olsder, 1999; Haurie et al., 2012; Başar et al., 2018). In the differential games literature, it is widely assumed that the players make their decisions at each instant of time or choose strategies that are piecewise continuous functions of time (also referred to as ordinary controls from here on). When one or more players choose actions only at certain specific time instants (also referred to as impulse controls from here on), the game problem is known as differential games with impulse controls. Zero-sum differential games where one player uses ordinary controls and the other uses impulse controls have been developed to study pursuit-evasion (Chikrii and Matichin, 2005; Chikrii et al., 2007), option pricing (El Farouq et al., 2010) and related problems. The strategic interactions taking place in pollution regulation, for instance, between a polluting firm and a regulator (Ferrari and Koch, 2019), and exchange rate management (Aïd et al., 2020) have been studied using two-player impulse differential games by considering that both players use impulse controls only.

The equilibrium of a differential game depends on the information that is available to the players when they make their decisions (Başar and Olsder, 1999). In the open-loop information structure, players' strategies depend on time and the initial state (a known parameter) while in the feedback information structure, players strategies' are functions of time and state values. A well-known result in the class of deterministic linear-state differential games (LSDGs) with ordinary controls is that open-loop Nash equilibria (OLNE) and feedback Nash equilibria (FNE) coincide (Dockner et al., 2000). This implies that a precommitment by the players to an action profile over time does not make them worse off than when they adapt their strategies to the state of the system. To the best of our knowledge, the literature does not provide a comparative analysis of open-loop and feedback Nash equilibria for differential games with impulse controls.

LSDGs have been extensively studied in the literature; see, e.g., Başar and Olsder (1999), Dockner et al. (2000), Engwerda (2005), Haurie et al. (2012). Their popularity stems from their tractability, that is, the equilibrium strategies and outcomes can be determined analytically. One drawback of this class of games is that, by definition, the model cannot include nonlinear terms in the state variables.[1] However, the fact that there is no restriction on the form of the control variables that enter the players' objective functionals or the dynamics renders LSDGs appealing in some applications of differential games (see Jørgensen and Zaccour, 2003). In this article, we consider a LSDG model with linear dynamics and quadratic cost functions for the players. The more general case can be obtained as an extension of our model by devising a numerical procedure to characterize the OLNE and FNE.

In this paper, we aim at (i) characterizing OLNE and FNE in LSDGs with impulse controls when the impulse instants are given; (ii) characterizing FNE when the impulse instants are endogenous (the open-loop case was studied in Sadana et al. (2021)); and (iii) verifying if OLNE and FNE coincide in LSDGs with impulse controls.

Our contributions are summarized as follows:

1. When the timing of impulses is fixed (or given exogenously), we provide analytical characterization of OLNE and FNE in Theorem 2.1 and Theorem 2.2, respectively. Further, we show in Theorem 2.3 that both equilibria coincide for this class of games.

2. When the number and timing of the impulses are also decision variables (or to be determined endogenously) of Player 2, besides the size of the impulse, we derive analytical expressions for OLNE in Theorem 2.4, and FNE in Theorem 2.5 and Theorem 2.7.

3. In the endogenous case, we show in Theorem 2.4 that the equilibrium number of impulses in the OLNE is at most three, whereas in the FNE, in Theorem 2.7, we show that there can be at most two impulses. In particular, when the instantaneous and terminal costs are both increasing or decreasing in state, we show that there can be at most one impulse in the feedback case, whereas there can be at most three impulses in the open-loop case. Moreover, we show that in the open-loop case, the equilibrium impulse timing of Player 2 depends on Player 1's problem parameters. In the feedback case, we show that such a dependency does not exist.

[1] It is possible to have a particular type of interaction between control and state variables and still retain the features of the class of LSDGs (see Dockner et al., 2000).

4. We provide generalization of our results for other cost structures in Theorem 2.8, and show that our results remain qualitatively unaltered for the multi-dimensional extension of our scalar LSDG model.

5. On the application side, we use our model to study the strategic decision making of two players, one of whom values the state positively and the other values the state negatively. To illustrate, we consider a firm (Player 1) that invests continuous effort to improve the security level of the system and the attacker (Player 2) exploits the system's vulnerabilities to lower its security.

This paper is organized as follows: In Section 2.1.1, we review the literature on impulse controls and differential games with impulse controls. In Section 2.2, we introduce our model. In Section 2.3, we compare the open-loop and feedback equilibria assuming that the impulse instants are known a priori while, in Section 2.4, we characterize the two equilibria when the impulse instants are endogenous. Further, in Section 2.5, we provide a numerical example to illustrate that OLNE and FNE differ in LSDGs when impulse instants are determined endogenously in the game. Some general results obtained by considering other cost structures and the multi-dimensional extension of our model are given in Section 2.6. Section 2.7 concludes.

2.1.1 Literature Review

In problems involving one decision maker, impulse controls have been quite naturally used in instances involving a fixed (or transaction) cost, as in, e.g., cash management (Baccarin, 2009), exchange rate intervention (Bertola et al., 2016), inventory control problems (Berovic and Vinter, 2004), demand throttling to manage server congestion (Perera et al., 2020), price management in retail energy markets (Basei, 2019), forest management (Alvarez, 2004), and investments in product innovation (Chahim et al., 2017). Some of the papers dealing with deterministic impulse controls include Berovic and Vinter (2004), Chahim et al. (2012, 2017), Leander et al. (2015), Reddy et al. (2016), and Grames et al. (2019).

Deterministic zero-sum differential games with impulse controls have been studied in Chikrii and Matichin (2005); Chikrii et al. (2007), El Farouq et al. (2010), and El Asri (2013). For stochastic zero-sum impulse-control differentiable games with one player using an ordinary control, and the other using an impulse control, see Azimzadeh (2019). In differential games with impulse control, the player who acts at discrete time instants solves an impulse control problem. The Hamiltonian Maximum Principle (see Blaquière, 1977a,b) and the Bensoussan-Lions quasi-variational inequalities (see Bensoussan and Lions, 1982, 1984) provide a framework to determine the time and level of such interventions. Recent works that use quasi-variational inequalities (QVIs) to determine the equilibrium in stochastic games with impulse control include Aïd et al. (2020) and Azimzadeh (2019). In a deterministic setting, QVIs are used in El Farouq et al. (2010).

The closest paper to our work is Aïd et al. (2020) where Nash equilibrium is obtained for stochastic nonzero-sum impulse games using the QVIs under the feedback information structure. However, they assumed that both players use threshold-type impulse controls only, that is, impulse controls are used when the state leaves the boundaries of a region. In contrast to their model, our game problem involves one player using ordinary controls and the other using impulse controls. Basei et al. (2019) study the N-person extension of the two-player game given in Aïd et al. (2020), and its corresponding mean field game. Aïd et al. (2020) also studied a LSDG model to derive analytical solutions.

Given that problems in regulation and cybersecurity (Taynitskiy et al., 2019) involve impulse controls, nonzero-sum differential games with impulse controls are useful for many diverse applications. Recently, Sadana et al. (2021) considered a class of finite-horizon two-player nonzero-sum linear-state differential games, where one player uses an ordinary control, while the other intervenes only at some instants of time in the game, that is, implements an impulse control. To illustrate, a game in which a firm continuously makes marketing, production, and security decisions, and a hacker attacks the firm occasionally fits the model in Sadana et al. (2021). When there are no fixed costs for Player 2 at the impulse instants and all the impulses are interior, i.e., impulse cannot occur at the initial and final time, Sadana et al. (2021) determined a unique OLNE using the Hamiltonian Maximum Principle. In this article, we determine both the OLNE and FNE by allowing for interior impulse instants, and also consider fixed costs in our model.[2] We also provide a comparative analysis of OLNE obtained us-

[2]A majority of applications of impulse controls consider fixed costs (see Cadenillas and Zapatero, 1999; Berovic and Vinter, 2004; Chahim et al., 2012, 2017; Bertola et al., 2016; Ferrari and Koch, 2019; Aïd et al., 2020).

ing Hamiltonian Maximum Principle and FNE derived from the QVIs for scalar deterministic nonzero-sum linear-state differential games with impulse controls.

2.2 Model

In this section, we introduce a scalar deterministic finite-horizon two-player nonzero-sum linear-state differential game model, where Player 1 uses ordinary controls while Player 2 uses impulse controls.

Let $T < \infty$ be the duration of the game. For Player 1, control action at time $t \in [0, T]$ is denoted by $u(t) \in \Omega_u \subset \mathbb{R}$, where Ω_u is a bounded and convex open subset of $\mathbb{R}$. We assume that $u : [0, T] \to \Omega_u$ is a piecewise continuous function of time and denotes the strategy profile of Player 1. The set of strategy profiles of Player 1 is denoted by $\mathcal{U}$. Player 2 intervenes or takes actions only at certain isolated time instants (or impulse instants) during the time period $[0, T]$. We denote by $\{\tau_1, \tau_2, \cdots, \tau_k\},\ k \in \mathbb{N}$ (the set of natural numbers), the set of intervention instants of Player 2, which satisfy the monotone increasing sequence property, that is,

$$0 \leq \tau_1 < \tau_2 < \cdots < \tau_k \leq T. \tag{2.1}$$

The state of the system evolves according to a scalar linear differential equation during the non-impulse instants of time as follows:

$$\dot{x}(t) = Ax(t) + Bu(t),\ t \neq \{\tau_1, \tau_2, \cdots, \tau_k\},\ x(0^-) = x_0, \tag{2.2}$$

where $x(t)$ denotes the state of the system at time $t \in [0, T]$, $x_0 \in \mathbb{R}$ denotes the initial state of the system, which is assumed to be given and 0^- denotes the time instant just before 0, and $A \in \mathbb{R}$ and $B \in \mathbb{R}\backslash\{0\}$ are constants. At the impulse instant τ_i ($i = 1, 2, \cdots, k$), Player 2 induces a jump in the state variable according to

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i, \tag{2.3}$$

where $v_i \in \Omega_v$ denotes the control action of Player 2 at impulse instant τ_i, and Ω_v denotes the control set of Player 2, which is assumed to be a bounded and convex open subset of $\mathbb{R}$. Here, $Q \in \mathbb{R}\backslash\{0\}$ is a constant.

The time instants before and after the impulse instant τ_i are denoted by τ_i^- and τ_i^+, respectively. Further, $x(\tau_i^-) = \lim_{t\uparrow\tau_i} x(t)$ and $x(\tau_i^+) = \lim_{t\downarrow\tau_i} x(t)$ are the state variables evaluated before and after the impulse instant τ_i. The strategy of Player 2 is denoted by $\tilde{v} := (\{(\tau_1, v_1), (\tau_2, v_2), \cdots, (\tau_k, v_k)\}, k) \in \mathcal{V}$, where $\mathcal{V}$ denotes the strategy set. We note that the number of impulses $k \in \mathbb{N}$ is also a decision variable of Player 2, where $k < \infty$. Clearly, Player 1 influences the evolution of the system during non-impulse instants (2.2) whereas Player 2's control results in jump in the state variable (2.3) at impulse instants.

Player 1 uses a strategy $u(.) \in \mathcal{U}$ to maximize the objective

$$J_1(x_0, u(.), \tilde{v}) = \int_0^T \frac{1}{2}\left(2w_1 x(t) + R_1 u(t)^2\right) dt + \sum_{i=1}^{k} q_1 x(\tau_i^-) + s_1 x(T^+), \tag{2.4}$$

where the integrand denotes the instantaneous payoff, the second term is the payoff received during the impulse instants, and the third term denotes the terminal payoff. T^+ denotes the time instant just after T. The parameters satisfy $w_1 \in \mathbb{R}$, $R_1 < 0$, $q_1 \in \mathbb{R}\backslash\{0\}$ and $s_1 \in \mathbb{R}$. Player 2 uses a strategy $\tilde{v} \in \mathcal{V}$ to maximize the objective

$$J_2(x_0, u(.), \tilde{v}) = \int_0^T w_2 x(t) dt + \sum_{i=1}^{k} \left(C + \frac{1}{2} P_2 v_i^2\right) + s_2 x(T^+), \tag{2.5}$$

where $C < 0$ denotes the fixed cost of each impulse and $\frac{1}{2}P_2 v_i^2$ the variable cost of the impulse at time instant τ_i, with $P_2 < 0$. Here, $w_2 \in \mathbb{R}$ and $s_2 \in \mathbb{R}$ are the instantaneous and terminal payoff parameters respectively. As the objectives of the players are interdependent, (2.2-2.5) describes a differential game with impulse controls. Further, as the objectives of the players as well as the dynamics are linear in the state variable, the game described by (2.2-2.5) is a linear-state differential game with impulse controls.

Remark 2.1 *Our main objective is to study the nature of the Nash equilibria when players' strategy spaces are different (piecewise continuous and discrete). The differential game model described by (2.2–2.5) is canonical, that is, minimal configuration required to capture the effect of differences in the strategy spaces. For this reason, we consider a two-player game with one player using piecewise continuous controls and the other player using impulse controls. Extension to $n > 2$ player case can be easily formulated with the framework studied in this paper.*

The Nash equilibrium strategies of the players are defined as follows:

Definition 2.1 *The strategy profile* $(u^*(.), \tilde{v}^*)$ *is a Nash equilibrium of the differential game (2.2–2.5) if the following inequalities are satisfied:*

$$J_1(x_0, (u^*(.), \tilde{v}^*)) \geq J_1(x_0, (u(.), \tilde{v}^*)),\ \forall u(.) \in \mathcal{U}, \tag{2.6a}$$

$$J_2(x_0, (u^*(.), \tilde{v}^*)) \geq J_2(x_0, (u^*(.), \tilde{v})),\ \forall \tilde{v} \in \mathcal{V}. \tag{2.6b}$$

In a differential game, the outcome varies with the information that is available to the players, when they take their decisions, also referred to as information structure; see Başar and Olsder (1999). Typically, two information structures are studied in the literature. In the open-loop information structure, players' strategies are functions of time and the initial state x_0, which is a known parameter. In our setting, this implies that Player 1's controls at time $t \in [0, T]$ are given by $u(t) := \gamma(t; x_0) \in \Omega_u$, where $\gamma : [0, T] \times \mathbb{R} \to \Omega_u$ is a measurable mapping. Similarly, the control action of Player 2 at an impulse instant $\tau_i \in [0, T]$ is given by $v_i := \delta(\tau_i; x_0) \in \Omega_v$, where $\delta : [0, T] \times \mathbb{R} \to \Omega_v$ is a measurable mapping. In the feedback information structure, the strategies of players are functions of time and the state variable. More precisely, Player 1's controls at time $t \in [0, T]$ are given by $u(t) := \gamma^f(t, x(t)) \in \Omega_u$, where $\gamma^f : [0, T] \times \mathbb{R} \to \Omega_u$ is a measurable mapping. Similarly, the control action of Player 2 at an impulse instant $\tau_i \in [0, T]$ is given by $v_i := \delta^f(\tau_i, x(\tau_i)) \in \Omega_v$, where $\delta^f : [0, T] \times \mathbb{R} \to \Omega_v$ is a measurable mapping.

Assumption 2.1 *The objective functions of Player 1 and Player 2 are strictly concave in their respective controls* $u(t) \in \Omega_u \subset \mathbb{R}$ *and* $v \in \Omega_v \subset \mathbb{R}$. *The interior of set* Ω_u *contains the equilibrium control of Player 1 and interior of* Ω_v *contains the equilibrium impulse level of Player 2.*

For bounded and convex open control sets, Assumption 2.1 is widely used to obtain the optimal controls using the first-order conditions, both in differential games (Başar and Olsder, 1999; Dockner et al., 2000) and in impulse control problems (Sobiesiak and Damaren, 2014; Chahim et al., 2012).

In the rest of the paper, we analyze two situations, first by treating the timing of the impulses of Player 2 as a problem parameter (or provided exogenously), and next as a decision variable (or occurs endogenously). In these two situations, we compare the Nash equilibria obtained under the open-loop and feedback information structures. To simplify the notations, we let $\tau_0 = 0$ and $\tau_{k+1} = T$ in the remainder of the paper.

2.3 Exogenous impulse instants

In this section, we consider the differential game (2.2-2.5), where the number of impulse instants k, and the timing of the impulse instants $\{\tau_1, \tau_2, \cdots, \tau_k\}$ are not decision variables of Player 2 but provided exogenously. So, the strategy of Player 2 is the set of control actions $\tilde{v} := \{v_1, v_2, \cdots, v_k\}$ to be taken at the given impulse instants $\{\tau_1, \tau_2, \cdots, \tau_k\}$. We characterize Nash equilibrium strategies for both open-loop and feedback information structures.

2.3.1 Open-loop Nash equilibrium

Computation of open-loop Nash equilibrium follows from (2.6a) and (2.6b). Let $(u^*(.), \tilde{v}^*)$ be the OLNE strategies of the players. From (2.6a), Player 1 solves an optimal control problem with additional costs, and jumps in the state variable at the impulse instants τ_i, $i = 1, 2, \cdots, k$, which make it a non-standard optimal control problem. The necessary conditions for optimality with jumps in the state variable and additional costs have been studied in the literature; see Geering (1976) and Sadana et al. (2021).[3] These conditions differ from those of classical optimal problem in that there is a jump in the co-state variable at the impulse instants. We define the Hamiltonian function of Player 1 as:

$$H_1(x(t), u(t), \lambda_1(t)) := w_1 x(t) + \frac{1}{2} R_1 u(t)^2 + \lambda_1(t)(Ax(t) + Bu(t)),$$

for $t \neq \{\tau_1, \tau_2, \cdots, \tau_k\}$, where $\lambda_1(t) \in \mathbb{R}$ is the co-state variable at time t. The necessary conditions are then given as follows: For $t \neq \{\tau_1, \tau_2, \cdots, \tau_k\}$,

$$u^*(t) = \arg\max_{u \in \Omega_u} H_1(x(t), u(t), \lambda_1(t)), \tag{2.7a}$$

and the state and co-state variables satisfy

$$\dot{x}(t) = H_{1\lambda_1}(x(t), u^*(t), \lambda_1(t)),\ x(0^-) = x_0, \tag{2.7b}$$

$$\dot{\lambda}_1(t) = -H_{1x}(x(t), u^*(t), \lambda_1(t)),\ \lambda_1(T^+) = s_1. \tag{2.7c}$$

[3] In Geering (1976), the authors assumed the state variable to be continuous and similar to Sadana et al. (2021), there are additional costs incurred at some exogenous time instants. In Sadana et al. (2021), the state variable is discontinuous, that is, $x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i^-), v_i^*)$, at the corresponding discrete time instants. Due to the state dependent jumps in the state variable and state dependent additional costs, the co-state variables satisfy $\lambda_1(\tau_i^-) = \lambda_1(\tau_i^+) + \frac{\partial}{\partial x}(q_1 x)\Big|_{x(\tau_i^-)} + \frac{\partial}{\partial x}(g(x, v_i^*))\Big|_{x(\tau_i^-)}$. Here $g(x, v_i^*) = Qv_i^*$ so we have $\frac{\partial}{\partial x}(g(x, v_i^*))\Big|_{x(\tau_i^-)} = 0$.

At the impulse instant τ_i $(i = 1, 2, \cdots, k)$, the jump in the state and co-state variables satisfy

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i^*, \tag{2.7d}$$

$$\lambda_1(\tau_i^-) = \lambda_1(\tau_i^+) + \frac{\partial}{\partial x}(q_1 x)\Big|_{x(\tau_i^-)} = \lambda_1(\tau_i^+) + q_1. \tag{2.7e}$$

The jump in the co-state equation (2.7e) is due to the state-dependent payoff accrued by Player 1 at the impulse instant τ_i.

Again from (2.6b), Player 2 solves an impulse optimal control problem with Player 1's strategies fixed at the Nash equilibrium strategy $u^*(.)$. The necessary conditions associated with an impulse optimal control problem were studied in the literature; see Blaquière (1977b) and Chahim et al. (2012). We introduce the Hamiltonian and impulse Hamiltonian functions as:

$$H_2(x(t), u(t), \lambda_2(t)) := w_2 x(t) + \lambda_2(t)(Ax(t) + Bu(t)), \tag{2.8a}$$

$$H_2^I(x(t), v_i, \lambda_2(t)) := C + \frac{1}{2}P_2 v_i^2 + \lambda_2(t) Q v_i, \tag{2.8b}$$

where $\lambda_2(t) \in \mathbb{R}$ denotes the co-state variable. The necessary conditions for optimality for Player 2's impulse optimal control problem are stated in the following lemma.

Lemma 2.1 *(Chahim et al., 2012, Theorem 2.2) Given the equilibrium controls $u^*(t)$ of Player 1 and the impulse instants $\{\tau_1, \tau_2, \cdots, \tau_k\}$, let $(x(t), v_1^*, v_2^*, \cdots, v_k^*)$ denote the optimal solution of the impulse control problem of Player 2. Then there exist co-states $\lambda_2(t) \in \mathbb{R}$ such that*
for $t \notin \{\tau_1, \tau_2, \ldots\ldots, \tau_k\}$,

$$\dot{x}(t) = Ax(t) + Bu^*(t), \; x(0^-) = x_0, \tag{2.9a}$$

$$\dot{\lambda}_2(t) = -H_{2x}(x(t), u^*(t), \lambda_2(t)), \; \lambda_2(T^+) = s_2, \tag{2.9b}$$

for $i = \{1, 2, \cdots, k\}$,

$$v_i^* = \arg\max_{v_i \in \Omega_v} H_2^I(x(\tau_i^-), v_i, \lambda_2(\tau_i^+)), \tag{2.9c}$$

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i^*, \tag{2.9d}$$

$$\lambda_2(\tau_i^-) = \lambda_2(\tau_i^+) + \frac{\partial}{\partial x}(H_2^I(x(t), v_i, \lambda_2(t)))\Big|_{x(\tau_i^-)} = \lambda_2(\tau_i^+).$$

Using (2.7) and (2.9), the next theorem characterizes the OLNE of the differential game described by (2.2-2.5).

Theorem 2.1 (Exogenous OLNE) *Let Assumption 2.1 hold. If the impulse instants* $\{\tau_1, \tau_2, \cdots, \tau_k\}$ *are given, then the unique OLNE strategies for* $A \neq 0$ *are given by*

$$u^*(t) = \frac{B}{R_1}\left(\frac{w_1}{A} - \left(\lambda_1(\tau_{j+1}^-) + \frac{w_1}{A}\right)e^{A(\tau_{j+1}^- - t)}\right), \quad \forall t \in (\tau_j, \tau_{j+1}),\ j \in \{0, 1, \cdots, k\}, \tag{2.10a}$$

$$v_i^* = \frac{Q}{P_2}\left(\frac{w_2}{A} - \left(s_2 + \frac{w_2}{A}\right)e^{A(T-\tau_i)}\right), \tag{2.10b}$$

where $i \in \{1, 2, \cdots, k\}$, $\lambda_1(\tau_{k+1}^+) = s_1$,

$$\lambda_1(t) = -\frac{w_1}{A} + \left(\lambda_1(\tau_{j+1}^-) + \frac{w_1}{A}\right)e^{A(\tau_{j+1}^- - t)}, \quad \forall t \in (\tau_j, \tau_{j+1}),\ j \in \{0, 1, \cdots, k\},$$

$$\lambda_1(\tau_i^-) = \lambda_1(\tau_i^+) + q_1,$$

so that, at the impulse instants, $\tau_i (i = 1, 2, \cdots, k)$*, we have*

$$\lambda_1(\tau_i^-) = -\frac{w_1}{A} + \left(\lambda_1(\tau_{i+1}^-) + \frac{w_1}{A}\right)e^{A(\tau_{i+1}^- - \tau_i^+)} + q_1.$$

For $A = 0$*, the unique OLNE strategies are given by*

$$u^*(t) = \frac{B}{R_1}\left(w_1(t - \tau_{j+1}^-) - \lambda_1(\tau_{j+1}^-)\right), \quad \forall t \in (\tau_j, \tau_{j+1}),\ j \in \{0, 1, \cdots, k\}, \tag{2.11a}$$

$$v_i^* = \frac{Q}{P_2}\left(w_2(\tau_i - T) - s_2\right), \tag{2.11b}$$

where $i \in \{1, 2, \cdots, k\}$, $\lambda_1(\tau_{k+1}^+) = s_1$,

$$\begin{aligned} &\lambda_1(t) = w_1(\tau_{j+1}^- - t) + \lambda_1(\tau_{j+1}^-),\ \forall t \in (\tau_j, \tau_{j+1}), j \in (0, 1, \cdots, k\}, \\ &\lambda_1(\tau_i^-) = \lambda_1(\tau_i^+) + q_1. \end{aligned} \tag{2.11c}$$

so that, at the impulse instants $\tau_i (i = 1, 2, \cdots, k)$*, we have*

$$\lambda_1(\tau_i^-) = w_1(\tau_{i+1}^- - \tau_i^+) + \lambda(\tau_{i+1}^-) + q_1.$$

Proof. Under Assumption 2.1 and from the optimality conditions for Player 1 and Player 2 given in (2.7a)–(2.7e) and (2.9a)–(2.9e), respectively, we can write the necessary conditions for OLNE as follows:

for $t \notin \{\tau_1, \tau_2, ..., \tau_k\}$,

$$u^*(t) = -\frac{B}{R_1}\lambda_1(t), \tag{2.12a}$$

$$\dot{x}(t) = Ax(t) - \frac{B^2}{R_1}\lambda_1(t),\ x(0^-) = x_0, \tag{2.12b}$$

$$\dot{\lambda}_1(t) = -A\lambda_1(t) - w_1,\ \lambda_1(T^+) = s_1, \tag{2.12c}$$

$$\dot{\lambda}_2(t) = -A\lambda_2(t) - w_2,\ \lambda_2(T^+) = s_2, \tag{2.12d}$$

for $i = \{1, 2, \cdots, k\}$,

$$v_i^* = -\frac{Q}{P_2}\lambda_2(\tau_i^+), \tag{2.12e}$$

$$x(\tau_i^+) = x(\tau_i^-) - \frac{Q^2}{P_2}\lambda_2(\tau_i^+), \tag{2.12f}$$

$$\lambda_1(\tau_i^-) = \lambda_1(\tau_i^+) + q_1, \tag{2.12g}$$

$$\lambda_2(\tau_i^-) = \lambda_2(\tau_i^+). \tag{2.12h}$$

From the above equations, we can obtain the expression for $\lambda_1(t)$ and $\lambda_2(t)$ as follows:
when $A \neq 0$:

$$\lambda_1(t) = -\frac{w_1}{A} + \left(\lambda_1(\tau_{j+1}^-) + \frac{w_1}{A}\right) e^{A(\tau_{j+1}^- - t)}, \text{ for } t \in (\tau_j, \tau_{j+1}),\ j \in \{0, 1, \cdots, k\}, \tag{2.13a}$$

$$\lambda_2(t) = -\frac{w_2}{A} + \left(s_2 + \frac{w_2}{A}\right) e^{A(T-t)}; \tag{2.13b}$$

when $A = 0$:

$$\lambda_1(t) = w_1(\tau_{j+1}^- - t) + \lambda_1(\tau_{j+1}^-),\ \forall\, t \in (\tau_j, \tau_{j+1}),\ j \in \{0, 1, \cdots, k\}, \tag{2.14a}$$

$$\lambda_2(t) = w_2(T - t) + s_2. \tag{2.14b}$$

On substituting the expressions of $\lambda_1(t)$ and $\lambda_2(t)$ for $A \neq 0$ and $A = 0$ in (2.12a) and (2.12e), respectively, we obtain the equilibrium controls of Player 1 and Player 2 given in (2.10a) and (2.10b) for $A \neq 0$, and (2.11a) and (2.11b) for $A = 0$. ■

2.3.2 Feedback Nash equilibrium

Feedback Nash equilibrium in the differential game (2.2)–(2.5) follows from (2.6a) and (2.6b), and can be obtained using dynamic programming. Before proceeding with the characterization of the FNE, we introduce the value function of Player 1, $V_1 : [0,T] \times \mathbb{R} \to \mathbb{R}$ and Player 2, $V_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$. From (2.6a), the value function V_1 is defined as follows:

$$V_1(t,x) = \max_{u(s),\ s\in[t,T]} \Big\{ \int_t^T \frac{1}{2}\left(2w_1 x(s) + R_1 u(s)^2\right) ds + \sum_{i=l}^{k} q_1 x(\tau_i^-) + s_1 x(T^+) \Big\}, \tag{2.15}$$

where the state variable evolves during the non-impulse instants $s \neq \{\tau_l, \tau_{l+1}, \cdots, \tau_k\}$, $t \leq \tau_l$, as

$$\dot{x}(s) = Ax(s) + Bu(s),\ x(t) = x,$$

and during a switching instant τ_i ($i = l, l+1, \cdots, k$) undergoes jumps according to

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i^*.$$

Similarly, following (2.6b), we define the value function associated with Player 2's impulse optimal control problem as follows:

$$V_2(t,x) = \max_{\{v_i\}_{i=1}^k} \Big\{ \int_t^T w_2 x(s) ds + \sum_{i=l}^{k} \left(C + \frac{1}{2}P_2 v_i^2\right) + s_2 x(T^+) \Big\}, \tag{2.16}$$

where the state variable evolves during the non-impulse instants $s \neq \{\tau_l, \tau_{l+1}, \cdots, \tau_k\}$, $t \leq \tau_l$, as

$$\dot{x}(s) = Ax(s) + Bu^*(s),\ x(t) = x,$$

and at a switching instant τ_i ($i = l, l+1, \cdots, k$) undergoes jumps according to

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i.$$

Given the linear-state structure of the differential game (2.2-2.5), we guess the form of the value functions of the players as follows:

Assumption 2.2 *The value functions of Player 1 and Player 2 are given by*

$$V_1(t,x) = m_1(t)x + n_1(t), \tag{2.17a}$$

$$V_2(t,x) = m_2(t)x + n_2(t). \tag{2.17b}$$

Next, using the dynamic programming principle, the FNE is characterized in the following theorem.

Theorem 2.2 (Exogenous FNE) *Let Assumption 2.1 and 2.2 hold. If the impulse instants*
$\{\tau_1, \tau_2, \cdots, \tau_k\}$ *are given, then the unique FNE for* $A \neq 0$ *is given by*

$$u^*(t) = \frac{B}{R_1}\left(\frac{w_1}{A} - \left(m_1(\tau_{j+1}^-) + \frac{w_1}{A}\right) e^{A(\tau_{j+1}^- - t)}\right), \quad \forall t \in (\tau_j, \tau_{j+1}), j \in \{0, 1, \cdots, k\}, \tag{2.18a}$$

$$v_i^* = \frac{Q}{P_2}\left(\frac{w_2}{A} - \left(s_2 + \frac{w_2}{A}\right) e^{A(T-\tau_i)}\right), \tag{2.18b}$$

where $i \in \{1, 2, \cdots, k\}$, $m_1(\tau_{k+1}^+) = s_1$,

$$\begin{aligned} m_1(t) &= -\frac{w_1}{A} + \left(m_1(\tau_{j+1}^-) + \frac{w_1}{A}\right) e^{A(\tau_{j+1}^- - t)}, \\ &\qquad \forall t \in (\tau_j, \tau_{j+1}), j \in \{0, 1, \cdots, k\}, \\ m_1(\tau_i^-) &= m_1(\tau_i^+) + q_1. \end{aligned} \tag{2.18c}$$

So, at the impulse instants, τ_i $(i \in \{1, 2, \cdots, k\})$, *we have*

$$m_1(\tau_i^-) = -\frac{w_1}{A} + \left(m_1(\tau_{i+1}^-) + \frac{w_1}{A}\right) e^{A(\tau_{i+1}^- - \tau_i^+)} + q_1.$$

For $A = 0$, *the unique FNE strategies are given by*

$$u^*(t) = \frac{B}{R_1}\left(w_1(t - \tau_{j+1}^-) - m_1(\tau_{j+1}^-)\right), \ \forall \ t \in (\tau_j, \tau_{j+1}), \ j \in \{0, 1, \cdots, k\}, \tag{2.19a}$$

$$v_i^* = \frac{Q}{P_2}\left(w_2(\tau_i - T) - s_2\right), \tag{2.19b}$$

where $i \in \{1, \cdots, k\}$, $m_1(\tau_{k+1}^+) = s_1$,

$$\begin{aligned} m_1(t) &= w_1(\tau_{j+1}^- - t) + m_1(\tau_{j+1}^-), \ \forall t \in (\tau_j, \tau_{j+1}), j \in \{0, 1, \cdots, k\}, \\ m_1(\tau_i^-) &= m_1(\tau_i^+) + q_1. \end{aligned} \tag{2.19c}$$

So, at the impulse instants, τ_i, $i \in \{1, 2, \cdots, k\}$, *we have*

$$m_1(\tau_i^-) = w_1(\tau_{i+1}^- - \tau_i^+) + m_1(\tau_{i+1}^-) + q_1.$$

Proof. See Appendix 2.8.1. ■

In the next theorem, we present the main result of this section that OLNE and FNE coincide in the differential games with impulse controls described by (2.2-2.5) when the impulse instants are given.

Theorem 2.3 *For the differential game described by (2.2–2.5), when the impulse instants $\{\tau_1, \tau_2, \cdots, \tau_k\}$ are fixed (or provided exogenously), and Assumption 2.1 and 2.2 hold, both OLNE and FNE coincide.*

Proof. Equation (2.10a) is structurally similar to (2.18a), and (2.11a) is structurally similar to (2.19a) because $\lambda_1(t)$ and $m_1(t)$ have the same dynamics, jump conditions and terminal conditions for $A \neq 0$ (see (2.10c) and (2.18c)) and $A = 0$ (see (2.11c) and (2.19c)). In particular, on replacing λ_1 with m_1 for $A = 0$ and for $A \neq 0$, we obtain that the OLNE and FNE strategies of Player 1 coincide. The OLNE and FNE strategies of Player 2 coincide because (2.10b) and (2.18b) hold true for $A = 0$, and (2.11b) and (2.19b) hold true for $A \neq 0$. ■

Remark 2.2 *Since the dynamic programming approach provides the sufficient conditions for Nash equilibria, and the FNE obtained by using the dynamic programming coincides with the OLNE obtained by using the necessary conditions, we have that the candidate OLNE are indeed the Nash equilibria.*

In the next section, we verify if the above result holds when the impulse timing is a decision variable of Player 2.

2.4 Endogenous impulse instants

In this section, we characterize the OLNE and FNE when the number and timing of impulse instants are part of Player 2's strategies (or occur endogenously). More importantly, we seek to investigate if both these informationally different equilibria also coincide in this case.

2.4.1 Open-loop Nash equilibrium

Let $(u^*(.), \tilde{v}^*)$ denote the open-loop Nash equilibrium strategy profile of the players. In particular, Player 2's equilibrium strategy is given by $\tilde{v}^* := (\{(\tau_1^*, v_1^*), (\tau_2^*, v_2^*), \cdots, (\tau_{k^*}^*, v_{k^*}^*)\}, k^*)$, where k^* and τ_i^* $(1 \leq i \leq k)$ denote the number and timing of impulses. From (2.6a), Player 1 solves an optimal control problem with Player 2's strategies fixed at the open-loop Nash equilibrium strategy $\tilde{v}^*$. This implies that the necessary conditions for optimality associated with Player 1's problem are also given by (2.7).

Concerning Player 2's impulse optimal control problem (2.6b), due to the presence of additional decision variables, that is, the number and timing of impulses, the necessary conditions for optimality differ from (2.9). In particular, additional consistency conditions are required to hold true at equilibrium impulse instants. These conditions follow from Chahim et al. (2012), and are summarized in the next lemma.

Lemma 2.2 *(Chahim et al., 2012, Theorem 2.2) Let the optimal solution of the impulse control problem of Player 2 be given by* $(\{(\tau_1^*, v_1^*), (\tau_2^*, v_2^*), \cdots, (\tau_{k^*}^*, v_{k^*}^*)\}, k^*)$*. Then there exist absolutely continuous functions* $\lambda_2 : [0^-, T^+] \to \mathbb{R}$*, with the Hamiltonian and impulse Hamiltonian functions defined in* (2.8a) *and* (2.8b)*, respectively, such that the following conditions hold true:*
for $t \notin \{\tau_1^*, \tau_2^*, \cdots, \tau_{k^*}^*\}$,

$$\dot{x}(t) = Ax(t) + Bu^*(t),\ x(0^-) = x_0, \tag{2.20a}$$

$$\dot{\lambda}_2(t) = -H_{2x}(x(t), u^*(t), \lambda_2(t)),\ \lambda_2(T^+) = s_2, \tag{2.20b}$$

and for $i = \{1, 2, \cdots, k^*\}$,

$$v_i^* = \arg\max_{v_i \in \Omega_v} H_2^I(x(\tau_i^{*-}), v_i, \lambda_2(\tau_i^+)), \tag{2.20c}$$

$$x(\tau_i^{*+}) = x(\tau_i^{*-}) + Qv_i^*, \tag{2.20d}$$

$$\lambda_2(\tau_i^{*-}) = \lambda_2(\tau_i^{*+}) + \frac{\partial}{\partial x}(H_2^I(x(t), v_i, \lambda_2(t)))\Big|_{x(\tau_i^{*-})} = \lambda_2(\tau_i^{*+}), \tag{2.20e}$$

$$H_2(x(\tau_i^{*+}), u^*(\tau_i^{*+}), \lambda_2(\tau_i^{*+})) - H_2(x(\tau_i^{*-}), u^*(\tau_i^{*-}), \lambda_2(\tau_i^{*-})) \begin{cases} > 0 & \textit{for } \tau_i^* = 0 \\ = 0 & \textit{for } \tau_i^* \in (0, T) \\ < 0 & \textit{for } \tau_i^* = T \end{cases} . \tag{2.20f}$$

Remark 2.3 *We note that* (2.20f) *is the additional consistency condition that is required to hold true when the number and timing of impulses are to be determined endogenously. The difference* $H_2(x(\tau_i^{*+}), u(\tau_i^{*+}), \lambda_2(\tau_i^{*+})) - H_2(x(\tau_i^{*-}), u(\tau_i^{*-}), \lambda_2(\tau_i^{*-}))$ *measures the gain made by Player 2 by delaying the impulse by one time instant (see Léonard and Long, 1992, Chapter 10).*

Remark 2.4 *In the characterization of the OLNE, we assume that Player 2 gives a nonzero impulse, that is, $v_i^* \neq 0$, at the equilibrium instants, $\tau_i^*, i \in \{1, 2, \cdots, k^*\}$. This assumption is justified because in this section, our objective is to show that OLNE and FNE differ when Player 2 decides the number and timing of impulse. Also, we shall see in the feedback case that the equilibrium impulse strategies involve nonzero equilibrium impulse levels.*

Using (2.7) and (2.20), we provide a characterization of the candidate OLNE in the next theorem. In the following discussion, to save on notation, we denote by $\delta := \left(\frac{P_2}{R_1}\right)\left(\frac{B}{Q}\right)^2$, then, as $P_2 < 0$ and $R_1 < 0$, we have $\delta > 0$.

Theorem 2.4 (Endogenous OLNE) *Let Assumption 2.1 hold, and let $w_2 \neq \delta q_1$ when $A = 0$. Then, the number of impulse instants for Player 2 is at most three, that is, $k^* \leq 3$, in the open-loop equilibrium. Further, when the parameters satisfy $w_2 \neq \delta q_1$, and either of the following conditions,*

$$T - \frac{1}{A}\ln\left(\frac{\delta q_1}{As_2 + w_2}\right) > 0, \tag{2.21a}$$

$$\frac{1}{A}\ln\left(\frac{\delta q_1}{As_2 + w_2}\right) > 0, \tag{2.21b}$$

then an interior impulse occurs in the time period $(0, T)$. For $A = 0$, there can be no interior impulse.
An impulse occurs at $\tau_{ol}^1 = 0$ if

$$\frac{(As_2 + w_2(e^{AT} - 1))(As_2 + w_2 e^{AT} - \delta q_1)}{A} > 0. \tag{2.21c}$$

An impulse occurs at $\tau_{ol}^2 = T$ if

$$s_2(As_2 - (\delta q_1 - w_2)) < 0. \tag{2.21d}$$

The equilibrium timing of interior impulse is given by

$$\tau_{ol}^I = T - \frac{1}{A}\ln\left(\delta\frac{q_1}{As_2 + w_2}\right). \tag{2.22}$$

With $k^* = 1$, the equilibrium control of Player 1 and equilibrium impulse levels of Player 2 are as follows:

For $\tau_{ol}^1 = 0$ *and* $t \in (0, T]$*, we have*

$$u_{ol}^*(t) = \begin{cases} \frac{B}{R_1}\left(\frac{w_1}{A} - (s_1 + \frac{w_1}{A})e^{A(T-t)}\right) & \textit{for } A \neq 0, \\ \frac{B}{R_1}(w_1(t-T) - s_1) & \textit{for } A = 0, \end{cases} \tag{2.23a}$$

$$v_{ol}^1 = \begin{cases} \frac{Q}{P_2}\left(\frac{w_2}{A} - (s_2 + \frac{w_2}{A})e^{AT}\right) & \textit{for } A \neq 0, \\ -\frac{Q}{P_2}(w_2 T + s_2) & \textit{for } A = 0. \end{cases} \tag{2.23b}$$

For $\tau_{ol}^I = T - \frac{1}{A}\ln\left(\delta\frac{q_1}{As_2 + w_2}\right)$*,* $A \neq 0$*,*

$$u_{ol}^*(t) = \begin{cases} \frac{B}{R_1}\left(\frac{w_1}{A} - (s_1 + \frac{w_1}{A})e^{A(T-t)}\right) & \textit{for } \tau_{ol}^I < t \leq T, \\ \frac{B}{R_1}\left(\frac{w_1}{A} - (s_1 + \frac{w_1}{A})e^{A(T-t)} - q_1 e^{A(\tau_{ol}{}^I - t)}\right) & \textit{for } 0 < t < \tau_{ol}^I, \end{cases} \tag{2.23c}$$

$$v_{ol}^I = \frac{Qw_2}{P_2 A} - \frac{B^2 q_1}{AQR_1}. \tag{2.23d}$$

For $\tau_{ol}^2 = T$ *and* $t \in [0, T)$*, we have*

$$u_{ol}^*(t) = \begin{cases} \frac{B}{R_1}\left(\frac{w_1}{A} - (s_1 + q_1 + \frac{w_1}{A})e^{A(T-t)}\right) & A \neq 0, \\ \frac{B}{R_1}(w_1(t-T) - s_1 - q_1) & A = 0, \end{cases} \tag{2.23e}$$

$$v_{ol}^2 = -\frac{Q}{P_2}s_2. \tag{2.23f}$$

Proof. From (2.7a) and Assumption 2.1, the first-order condition gives the equilibrium control of Player 1

$$u^*(t) = -\frac{B}{R_1}\lambda_1(t).$$

When Player 2 solves her optimal control problem (with Player 1's strategy fixed at her OLNE strategy), conditions (2.20a)–(2.20f) hold true. From (2.20c) and Assumption 2.1, we get the equilibrium impulse level as follows:

$$v_i^* = -\frac{Q}{P_2}\lambda_2(\tau_i^+).$$

From (2.20b) and (2.20e), the co-state $\lambda_2(t)$ is given by

$$\lambda_2(t) = \begin{cases} -\frac{w_2}{A} + \left(s_2 + \frac{w_2}{A}\right)e^{A(T-t)}, & A \neq 0 \\ w_2(T-t) + s_2, & A = 0. \end{cases}$$

Now, we determine the candidates for the equilibrium impulse instant. First, we analyze the situation where the equilibrium impulse instant satisfies $\tau_i^* \in (0, T)$. Following the Hamiltonian continuity condition (2.20f) at $\tau_i^* \in (0, T)$, we have

$$w_2 x(\tau_i^{*+}) + \lambda_2(\tau_i^{*+})(Ax(\tau_i^{*+}) + Bu(\tau_i^{*+})) = w_2 x(\tau_i^{*-}) + \lambda_2(\tau_i^{*-})(Ax(\tau_i^{*-}) + Bu(\tau_i^{*-})),$$

Substituting $u^*(t)$ in the above equation, and using the conditions, (2.7e), (2.20d), (2.20e), we obtain

$$-\frac{Q^2}{P_2}\left(A\lambda_2(\tau_i^*) + (w_2 - \delta q_1)\right)\lambda_2(\tau_i^*) = 0. \tag{2.24}$$

Next, we provide a justification for the assumption $w_2 \neq \delta q_1$ when $A = 0$. Assume that $w_2 = \delta q_1$, then the above condition results in $A\lambda_2^2(\tau_i^*) = 0$. If $A = 0$, then (2.24) holds true at all $\tau^* \in (0, T)$. From the isolated property of the impulse instants (2.1), this is not possible.

When $A = 0$, and as $(w_2 - \delta q_1) \neq 0$, (2.24) results in $\lambda_2(\tau_i^*) = 0$, and this contradicts the occurrence of impulse at $\tau_i^* \in (0, T)$. So, there is no interior impulse when $A = 0$ since we have assumed that for admissible equilibrium impulse instants, $v_i \neq 0$.

When $A \neq 0$ and $w_2 = \delta q_1$, we have that $\lambda_2(\tau_i^*) = 0$. This implies that $v_i^* = 0$, which contradicts the idea that impulse occurs at $\tau_i^* \in (0, T)$. So, an impulse does not occur in $(0, T)$ when $A \neq 0$ and $w_2 = \delta q_1$.

When $A \neq 0$, (2.24) can be written as

$$A\left(\lambda_2(\tau_i^*) - \frac{\delta q_1 - w_2}{A}\right)\lambda_2(\tau_i^*) = 0.$$

This implies that the impulse instant is characterized by $\lambda_2(\tau_i^*) = \frac{\delta q_1 - w_2}{A}$. From (2.20b), we have $\lambda_2(t) = -\frac{w_2}{A} + \left(s_2 + \frac{w_2}{A}\right)e^{A(T-t)}$ for all $t \in [0, T]$. As, the co-state function $\lambda_2 : [0, T] \to \mathbb{R}$ is strictly monotone, we have at most one impulse instant $\tau_i^* \in (0, T)$ that solves the equation

$$\lambda_2(\tau_i^*) = -\frac{w_2}{A} + \left(s_2 + \frac{w_2}{A}\right)e^{A(T-\tau_i^*)} = \frac{\delta q_1 - w_2}{A}.$$

The unique interior equilibrium impulse instant denoted by τ_{ol}^I is given by

$$\tau_{\text{ol}}^I = T - \frac{1}{A}\ln\left(\left(\frac{B}{Q}\right)^2\left(\frac{P_2}{R_1}\right)\frac{q_1}{As_2 + w_2}\right)$$

$$= T - \frac{1}{A}\ln\left(\frac{\delta q_1}{As_2 + w_2}\right). \tag{2.25}$$

Since $\tau_{\text{ol}}^I \in (0, T)$, we must have (2.21a)-(2.21b) which are expressed in terms of problem parameters.

Next, if there is an impulse at the initial time, then from (2.20f), (2.7e), (2.20d), (2.20e), we have

$$\lambda_2(0)\left(A\lambda_2(0) - (\delta q_1 - w_2)\right) > 0.$$

On substituting $\lambda_2(0) = \frac{As_2 e^{AT} + w_2(e^{AT}-1)}{A}$, we get inequality (2.21c) that describes the problem parameters when impulse occurs at the initial time.

Next, if there is an impulse at the final time, then from (2.20f), (2.7e), (2.20d), (2.20e), we have

$$\lambda_2(T)\left(A\lambda_2(T) - (\delta q_1 - w_2)\right) < 0,$$

On substituting $\lambda_2(T) = s_2$, we find that an impulse occurs at the final time when (2.21d) holds true.

Using (2.7c) and (2.7e), we obtain the co-state variable $\lambda_1(t)$ satisfies (2.12c) and (2.12g) at the impulse instants. With $k^* = 1$ and impulses at $t = 0$, $t = \tau_{\text{ol}}^I$, $t = T$, the equilibrium controls of Player 1 and the equilibrium impulse levels of Player 2 are given by (2.23). ■

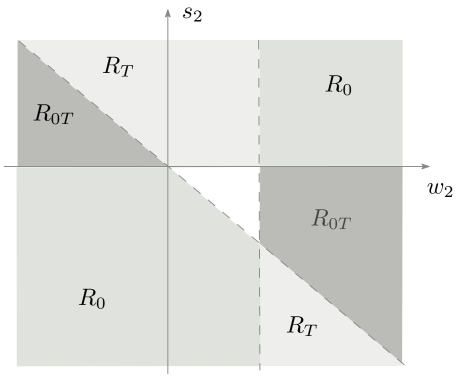
Remark 2.5 *In Theorem 2.4, we have only provided the equilibrium controls of the players when $k^* = 1$ for brevity. The equilibrium controls of the players for $k^* = 2$ and $k^* = 3$ can be obtained by using the necessary conditions* (2.7) *and* (2.20).

Remark 2.6 *Since the continuous Hamiltonian of Player 2 is a function of the equilibrium control of Player 1, the impulse timing also depends on the problem parameters of Player 1.*

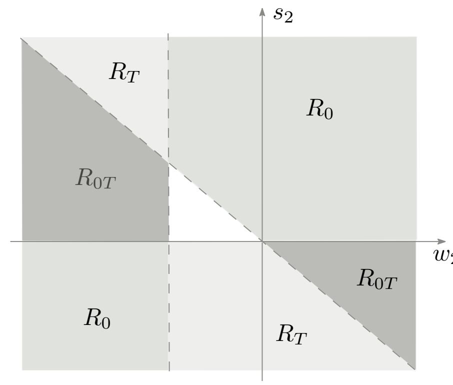
The parameter values which satisfy the inequalities (2.21a)–(2.21b), (2.21c), (2.21d) are shown in Figure 2.1.

2.4.2 Feedback Nash equilibrium

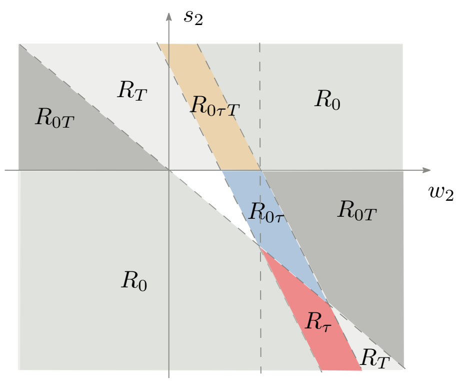
Next, we characterize the FNE when both the level and timing of the impulse instants are Player 2's decision variables. First, we consider Player 1's optimal control problem assuming that Player 2's equilibrium policy $\tilde{v}^* = \{(\tau_1^*, v_1^*), (\tau_2^*, v_2^*),$



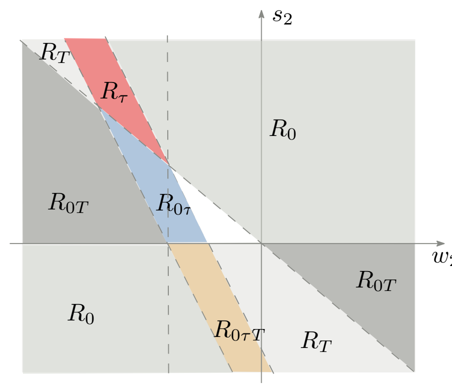
(a) $A = 0,\ q > 0$



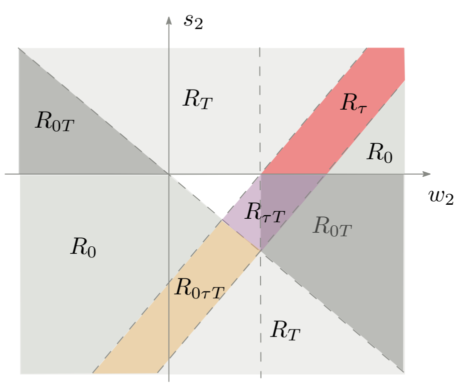
(b) $A = 0,\ q < 0$



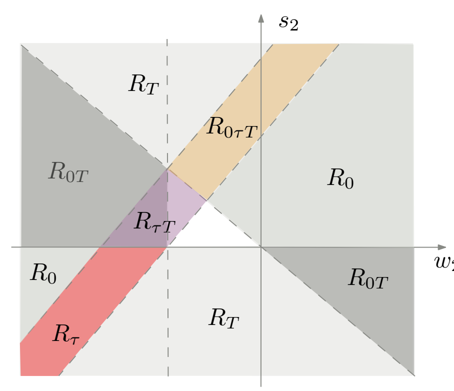
(c) $A > 0,\ q > 0$



(d) $A > 0,\ q < 0$



(e) $A < 0,\ q > 0$



(f) $A < 0,\ q < 0$

Figure 2.1 – The regions are described as follows: R_0 : Impulse at $t = 0$, R_T : Impulse at $t = T$, R_{0T} : Impulse at $t = 0$ and $t = T$, $R_{0\tau}$: Impulse at $t = 0$ and $t = \tau_{ol}^I$, $R_{\tau T}$: Impulse at $t = \tau_{ol}^I$ and $t = T$, $R_{0\tau T}$: Impulse at t=0, $t = \tau_{ol}^I$ and $t = T$.

$\cdots,(\tau_k^*, v_{k^*}^*), k^*\}$ is given. Similar to the analysis done in Section 2.3.2, let $V_1 : [0,T]\times\mathbb{R}\to\mathbb{R}$ denote the value function of Player 1. Then, we have

$$V_1(t,x) = \max_{u(s),\ s\in[t,T]} \Big\{ \int_t^T \frac{1}{2}\left(2w_1 x(s) + R_1 u(s)^2\right) ds + \sum_{i=l}^{k^*} q_1 x(\tau_i^{*-}) + s_1 x(T^+)\Big\}, \tag{2.26}$$

where the state variable evolves during the non-impulse instants $s \neq \{\tau_l^*, \tau_{l+1}^*, \cdots, \tau_{k^*}^*\}$, $t \leq \tau_l^*$ as

$$\dot{x}(s) = Ax(s) + Bu(s),\ x(t) = x,$$

and during a switching instant τ_i^* ($i = 1, 2, \cdots, k^*$) undergoes jumps according to

$$x(\tau_i^{*+}) = x(\tau_i^{*-}) + Qv_i^*.$$

In the impulse-free region $[\tau_i^{*+}, \tau_{i+1}^{*-}]$, the following Hamilton-Jacobi-Bellman (HJB) equation holds true:

$$-\frac{\partial V_1(t,x)}{\partial t} = \max_{u\in\Omega_u}\left(w_1 x + \frac{1}{2}R_1 u(t)^2 + \left(\frac{\partial V_1}{\partial x}\right)(Ax + Bu(t))\right). \tag{2.27}$$

At the jump instants, $\{\tau_1^*, \tau_2^*, \cdots, \tau_{k^*}^*\}$, the value functions are related as follows:

$$V_1(\tau_i^{*-}, x(\tau_i^{*-})) = V_1(\tau_i^{*+}, x(\tau_i^{*+})) + q_1 x(\tau_i^{*-}). \tag{2.28}$$

Given the equilibrium strategy $u^*(.)$ of Player 1, following (2.6b), we define the value function associated with Player 2's impulse optimal control problem as follows:

$$V_2(t,x) = \max_{\{(\tau_i,v_i)_{i=1}^k\}} \Big\{ \int_t^T w_2 x(s) ds + \sum_{i=1}^{k}\left(C + \frac{1}{2}P_2 v_i^2\right) + s_2 x(T^+)\Big\}, \tag{2.29}$$

where the state variable evolves during the non-impulse instants $s \neq \{\tau_1, \tau_2, \cdots, \tau_k\}$, as

$$\dot{x}(s) = Ax(s) + Bu^*(s),\ x(t) = x,$$

and during a switching instant τ_i ($i = 1, 2, \cdots, k$) undergoes jumps according to

$$x(\tau_i^+) = x(\tau_i^-) + Qv_i.$$

We emphasize that Player 2's problem differs, structurally, in the endogenous case from the exogenous case as the number of impulses k and the timing of the impulses τ_i $(i = 1, 2, \cdots, k)$ are also decision variables to be determined besides the size of the impulses v_i $(i = 1, 2, \cdots, k)$. Impulse optimal control problems with endogenous decision variables are closely related to optimal stopping problems, and use tools from QVIs; see Bensoussan and Lions (1982) , Bensoussan and Tapiero (1982), and Bensoussan and Lions (1984) for early works in this area. In the following discussion, we briefly summarize the necessary concepts associated with QVIs before proceeding with the characterization of the FNE.

Assumption 2.3 *The value function $V_2 : [0, T] \times \mathbb{R} \to \mathbb{R}$ is continuous and continuously differentiable in its arguments.*

Given the value function $V_2(t, x)$ of Player 2, we define the operator $\mathcal{R}$ as follows:

$$\mathcal{R}V_2(t, x) := \max_{v \in \Omega_v} \left(\frac{1}{2} P_2 v^2 + C + V_2(t, x + Qv) \right). \tag{2.30}$$

We introduce the Hamiltonian function $\mathcal{H}_2 : [0, T] \times \mathbb{R} \times \mathbb{R}$ as follows:[4]

$$\mathcal{H}_2(x, t, \frac{\partial V_2}{\partial x}) = w_2 x + \frac{\partial V_2}{\partial x} \left(Ax + Bu(t) \right). \tag{2.31}$$

From Aubin (1982) and Bensoussan and Lions (1982, 1984), it can be shown that the value function (2.29) satisfies the following Bensoussan-Lions quasi-variational inequalities, that is,

$$\frac{\partial V_2}{\partial t} + \mathcal{H}_2(x, t, \frac{\partial V_2}{\partial x}) \leq 0, \ \forall (t, x) \in (0, T) \times \mathbb{R}, \tag{2.32a}$$

$$V_2(t, x) - \mathcal{R}V_2(t, x) \geq 0, \ \forall (t, x) \in [0, T] \times \mathbb{R}, \tag{2.32b}$$

$$\left(\frac{\partial V_2}{\partial t} + \mathcal{H}_2(x, t, \frac{\partial V_2}{\partial x}) \right) \left(V_2(t, x) - \mathcal{R}V_2(t, x) \right) = 0, \forall (t, x) \in (0, T) \times \mathbb{R}, \tag{2.32c}$$

$$V_2(T, x) = \max\{\zeta(x), s_2 x\}, \tag{2.32d}$$

$$\text{where } \zeta(x) = \max_{v \in \Omega_v} \{ s_2(x + Qv) + C + \frac{1}{2} P_2 v^2 \}. \tag{2.32e}$$

In the following, we provide a heuristic interpretation of the QVIs (2.32). When the state is at a given level x at time t, Player 2 can either give an impulse or wait.

[4]The Hamiltonian associated with the value function of Player 2 is different from the Hamiltonian of Player 2 given in (2.8a) associated with the co-state of Player 2. The two Hamiltonians are equal when the gradient of the value function is equal to the co-state variable.

Suppose that an impulse does not occur in the time interval $[t, t+h]$. Since Player 2 waits, using the dynamic programming principle, we conclude that the value function is bounded from below by the sum of the running profit from t to $t+h$ and the optimal profit from time $t+h$ onwards, that is,

$$V_2(t,x) \geq \int_t^{t+h} w_2 x(s) ds + V_2(t+h, x(t+h)).$$

From Assumption 2.3 and using a Taylor series expansion of the above expression, and letting $h \to 0$, we obtain (2.32a). If it is optimal for Player 2 to give an impulse at time t, then the state jumps from $x(t)$ to $x(t) + Qv$, such that

$$V_2(t,x) \geq \max_{v \in \Omega_v} \left(C + \frac{1}{2} P_2 v^2 + V_2(t, x+Qv) \right) =: \mathcal{R}V_2(t,x).$$

This verifies (2.32b). Clearly, at any (t,x), Player 2 can either wait, which implies that (2.32a) holds with equality, or she can give an impulse so that (2.32b) holds with equality. This implies that the complementarity condition (2.32c) holds to ensure that either (2.32a) or (2.32b) holds with equality. If there is no impulse at the final time, the value function is equal to the salvage value; otherwise, the value function is equal to the maximum value that Player 2 can obtain by giving an impulse at T, and this justifies condition (2.32d).

Using (2.30), we define the following two sets. The first is a stopping or intervention set $\mathcal{S}$, which is defined as

$$\mathcal{S} := \left\{ (t,x) \in [0,T] \times \mathbb{R} \;\middle|\; V_2(t,x) = \mathcal{R}V_2(t,x) \right\}. \tag{2.33}$$

The stopping set characterizes all the data points $(t,x) \in [0,T] \times \mathbb{R}$ where it is optimal for Player 2 to give an impulse. The second is a continuation set $\mathcal{C}$, defined as

$$\mathcal{C} := \left\{ (t,x) \in [0,T] \times \mathbb{R} \;\middle|\; V_2(t,x) > \mathcal{R}V_2(t,x) \right\}. \tag{2.34}$$

Clearly, from the definition of $\mathcal{C}$, it is optimal for Player 2 to not give an impulse at the data point $(t,x) \in \mathcal{C}$. In other words, the continuation set characterizes the impulse-free region.

It is well known that finding a function that satisfies QVIs is a difficult problem, see Cadenillas and Zapatero (1999), Bertola et al. (2016). Therefore, analytical solutions have been obtained by making regularity assumptions on the value function and impulse controls. Though the system of QVIs can be shown to be sufficient conditions for impulse control problems under less restrictive assumptions on the value functions (Berovic and Vinter, 2004) than those made in Assumption 2.3, our objective in this section is to verify if the classical result, that open-loop and feedback equilibria coincide in deterministic LSDGs, holds when Player 2 decides the timing of impulses.

Given the linear structure of the game in the state variable, our conjecture (see Dockner et al., 2000; Başar et al., 2018) on the value functions of Player 1 and Player 2 that satisfy the HJB equation (2.27) and QVIs (2.32a-2.32d), respectively, is given in the form of the following assumption:

Assumption 2.4 *The value function of Player i $(i = 1, 2)$ is given by*

$$V_i(t, x) = \alpha_i(t)x + \beta_i(t). \tag{2.35}$$

Using a linear value function for the case with exogenous impulse instants, we showed in Section 2.3 that open-loop and feedback Nash equilibrium coincide. Next, we show that the value functions given in Assumption 2.4 indeed solve the system of QVIs. The next theorem characterizes the impulse instants in the FNE when the impulse timing is endogenously determined by Player 2. To save on notation, we introduce $\gamma = \sqrt{\frac{2P_2C}{Q^2}}$.

Theorem 2.5 *Let Assumption 2.1 and 2.4 hold. Let $As_2 + w_2 \neq 0$ when $s_2 = \gamma$ or $s_2 = -\gamma$. There can be at most two impulses in the FNE, and they occur at $\tau^1_{fb} = 0$ and $\tau^2_{fb} = T$.*

Proof. We substitute the value function of Player 1 given in Assumption 2.4 in the HJB equation (2.27) to obtain

$$-\dot{\alpha}_1(t)x - \dot{\beta}_1(t) = \max_{u(t)\in\Omega_u} \{w_1 x + \frac{1}{2}R_1 u(t)^2 + \alpha_1(t)(Ax + Bu(t))\}.$$

Following Assumption 2.1 on the interior solutions, the first-order condition associated with the above maximization problem results in

$$u^*(t) = -\frac{B}{R_1}\frac{\partial V_1(t, x)}{\partial x} = -\frac{B\alpha_1(t)}{R_1}. \tag{2.36}$$

Substituting for the above solution in the HJB equation (2.27), we obtain

$$-\dot{\alpha}_1(t)x - \dot{\beta}_1(t) = w_1 x - \frac{B^2\alpha_1(t)^2}{2R_1} + A\alpha_1(t)x.$$

Applying the method of undetermined coefficients gives

$$\dot{\alpha}_1(t) = -w_1 - A\alpha_1(t),\ \alpha_1(T^+) = s_1, \tag{2.37a}$$

$$\dot{\beta}_1(t) = \frac{B^2\alpha_1(t)^2}{2R_1},\ \beta_1(T^+) = 0. \tag{2.37b}$$

From (2.28), at any impulse instant τ_i^*, we have the following relation:

$$\alpha_1(\tau_i^{*-})x(\tau_i^{*-}) + \beta_1(\tau_i^{*-}) = \alpha_1(\tau_i^{*+})(x(\tau_i^{*-}) + Qv_i^*) + \beta_1(\tau_i^{*+}) + q_1 x(\tau_i^{*-}),$$

which implies

$$\alpha_1(\tau_i^{*-}) = \alpha_1(\tau_i^{*+}) + q_1, \tag{2.38a}$$

$$\beta_1(\tau_i^{*-}) = \beta_1(\tau_i^{*+}) + \alpha_1(\tau_i^{*+})Qv_i^*. \tag{2.38b}$$

Next, we determine the coefficients of the value function of Player 2 given in Assumption 2.4. First, we determine the values of $\alpha_2(T)$ and $\beta_2(T)$. Following Assumption 2.1, we take the partial derivative of the right-hand side of (2.32e) with respect to v and equate it to 0 to obtain $v_T^* = -\frac{s_2 Q}{P_2}$. Substituting v_T^* in (2.32e), we obtain

$$\zeta(x) = s_2 x + C - \frac{(s_2 Q)^2}{2P_2}. \tag{2.39}$$

Clearly, $\zeta(x) \geq s_2 x$ if $\frac{2P_2C}{Q^2} \leq s_2^2$. From (2.32d), we obtain that if $\frac{2P_2C}{Q^2} \leq s_2^2$,

$$\alpha_2(T)x + \beta_2(T) = s_2 x + C - \frac{(s_2 Q)^2}{2P_2},$$

$$\Rightarrow \alpha_2(T) = s_2,\ \beta_2(T) = C - \frac{(s_2 Q)^2}{2P_2}, \tag{2.40}$$

and if $\frac{2P_2C}{Q^2} \geq s_2^2$, then

$$\alpha_2(T)x + \beta_2(T) = s_2 x,$$

$$\Rightarrow \alpha_2(T) = s_2,\ \beta_2(T) = 0. \tag{2.41}$$

In the impulse-free region, (2.32a) holds with equality. Using (2.36) in (2.32a), we obtain

$$w_2 x(t) + \dot{\alpha}_2(t)x + \dot{\beta}_2(t) + \alpha_2(t)\left(Ax - \frac{B^2\alpha_1(t)}{R_1}\right) = 0.$$

Applying the method of undetermined coefficients gives

$$\dot{\alpha}_2(t) = -w_2 - A\alpha_2(t), \alpha_2(T) = s_2, \tag{2.42a}$$

$$\dot{\beta}_2(t) = \frac{B^2\alpha_1(t)\alpha_2(t)}{R_1}, \tag{2.42b}$$

where $\beta_2(T)$ is given by (2.40) if there is an impulse at T, and if it is not optimal to give an impulse then $\beta_2(T)$ is given by (2.41). Solving for $\alpha_2(t)$, we have

$$\alpha_2(t) = w_2(T-t) + s_2,\ A = 0, \tag{2.43a}$$

$$\alpha_2(t) = -\frac{w_2}{A} + e^{A(T-t)}\left(s_2 + \frac{w_2}{A}\right),\ A \neq 0. \tag{2.43b}$$

Under Assumption 2.4, we compute $\mathcal{R}V_2$ as

$$\mathcal{R}V_2(t,x) = \max_{v\in\Omega_v}\{C + \frac{1}{2}P_2v^2 + \alpha_2(t)(x+Qv) + \beta_2(t)\}. \tag{2.44}$$

Following Assumption 2.1 on the interior solutions, the first-order condition associated with the maximization problem (2.44) results in

$$v^* = -\frac{Q\alpha_2(t)}{P_2}. \tag{2.45}$$

Substituting the above solution in (2.44) yields

$$\mathcal{R}V_2(t,x) = C + \frac{Q^2\alpha^2(t)}{2P_2} + V_2(t,x) \tag{2.46}$$

$$\Rightarrow V_2(t,x) - \mathcal{R}V_2(t,x) = -C + \frac{Q^2\alpha^2(t)}{2P_2}. \tag{2.47}$$

Then, the stopping set (2.33) is given by

$$\mathcal{S} := \left\{(t,x) \in [0,T]\times\mathbb{R} \;\middle|\; \alpha_2^2(t) = \frac{2P_2C}{Q^2}\right\}, \tag{2.48}$$

and the continuation set (2.34) is given by

$$\mathcal{C} := \left\{(t,x) \in [0,T]\times\mathbb{R} \;\middle|\; \alpha_2^2(t) < \frac{2P_2C}{Q^2}\right\}. \tag{2.49}$$

When $As_2 + w_2 = 0$, there is an impulse at each instant of time for $s_2 = \gamma$ and for $s_2 = -\gamma$ which means that the impulse instants do not satisfy monotone increasing sequence property given in (2.1). From (2.43a) and (2.43b), we know that for $As_2 + w_2 = 0$, $\alpha_2(t) = s_2$. So, for $As_2 + w_2 = 0$, there is no impulse when $s_2 \neq \gamma$ and $s_2 \neq -\gamma$. Next, we analyze the cases where $As_2 + w_2 \neq 0$.

Clearly, $\alpha_2(t)$ given in (2.43a) and (2.43b) is strictly monotone in t for $As_2 + w_2 \neq 0$, so it can take values $\sqrt{\frac{2P_2C}{Q^2}}$ and $-\sqrt{\frac{2P_2C}{Q^2}}$ at most once. This naturally implies from equation (2.49), that there can be at most two impulses, and they occur at $\tau_{\text{fb}}^1 = 0$ and $\tau_{\text{fb}}^2 = T$. ■

Remark 2.7 *In the linear-state differential games with impulse control, the stopping set given in* (2.48)*, and the continuation set given in* (2.49) *are independent of the state of the system.*

From (2.32b), the value function must satisfy $V_2(t,x) \geq \mathcal{R}V_2(t,x) \Rightarrow \alpha_2^2(t) \leq \gamma^2$ for all $(t,x) \in [0,T] \times \mathbb{R}$. As a result, this condition imposes certain restrictions on the parameter region where the linear value function is well-defined. Next theorem characterizes this region.

Theorem 2.6 *Let Assumption 2.4 hold true. Let $As_2 + w_2 \neq 0$ when $s_2 = \gamma$ or $s_2 = -\gamma$. The linear value function* (2.35) *is well-defined when the parameters satisfy the following conditions.*

(i) $A = 0$, $w_2 \geq 0$, $Tw_2 + s_2 \leq \gamma$, $s_2 \geq -\gamma$

(ii) $A = 0$, $w_2 \leq 0$, $Tw_2 + s_2 \geq -\gamma$, $s_2 \leq \gamma$

(iii) $A > 0$, $As_2 + w_2 > 0$, $s_2 \geq -\gamma$, $As_2 e^{AT} + w_2(e^{AT} - 1) - A\gamma \leq 0$

(iv) $A > 0$, $As_2 + w_2 < 0$, $s_2 \leq \gamma$, $As_2 e^{AT} + w_2(e^{AT} - 1) + A\gamma \geq 0$

(v) $A < 0$, $As_2 + w_2 > 0$, $s_2 \geq -\gamma$, $As_2 e^{AT} + w_2(e^{AT} - 1) - A\gamma \geq 0$

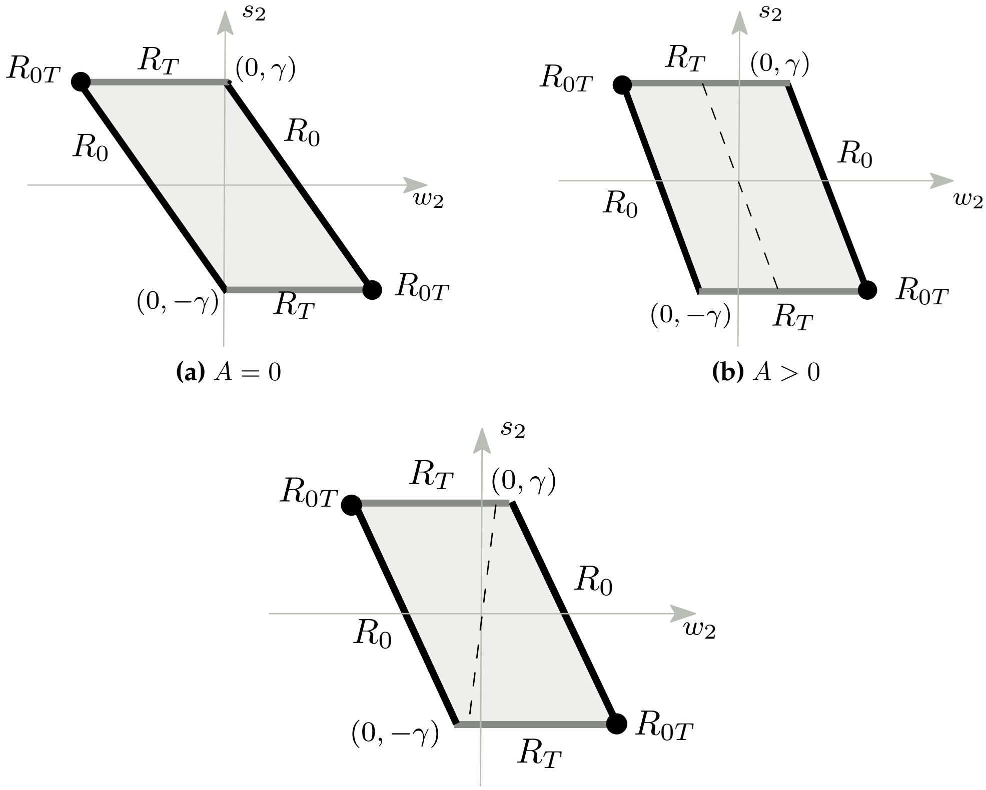
(vi) $A < 0$, $As_2 + w_2 < 0$, $s_2 \leq \gamma$, $As_2 e^{AT} + w_2(e^{AT} - 1) + A\gamma \leq 0$

Proof. We recall that the value function $V_2(t,x)$ must satisfy the condition (2.32b). This implies $\alpha_2^2(t) \leq \gamma^2$ for all $t \in [0,T]$.

With $A = 0$, we get $\alpha_2(t) = w_2(T-t) + s_2$, which is an increasing (decreasing) function of time t when w_2 is negative (positive). Then, we must have $(w_2(T-t) + s_2)^2 \leq \gamma^2$ for all $t \in [0,T]$, and this condition is satisfied when conditions (i)–(ii) hold true.

When $A \neq 0$, we get $\alpha_2(t) = -\frac{w_2}{A} + e^{A(T-t)}\left(s_2 + \frac{w_2}{A}\right)$ is decreasing in t if $As_2 + w_2 > 0$ and is increasing in t if $As_2 + w_2 < 0$. Using a similar analysis as before, for $A > 0$ and $A < 0$, the value function is defined only in the region where the parameters satisfy $\left(-\frac{w_2}{A} + e^{A(T-t)}\left(s_2 + \frac{w_2}{A}\right)\right)^2 \leq \gamma^2$, which is characterized by the conditions (iii)–(iv) and (v)–(vi). ■

The parameter regions where the value function $V_2 : [0,T] \times \mathbb{R} \to \mathbb{R}$ is well-defined is illustrated in the Figure 2.2. In particular, the shaded regions in the



(c) $A < 0$

Figure 2.2 – Shaded regions correspond to parameter space in the (w_2, s_2) plane for which the value function is well-defined. An impulse occurs at $t = T$ for parameters corresponding to the upper and lower boundaries of the shaded regions, denoted by R_T. For the left and right boundaries denoted by R_0, there is an impulse at $t = 0$. R_{0T} denotes that impulse occur at $t = 0$ and $t = T$.

Figures 2.2a, 2.2b and 2.2c correspond to the regions defined by the conditions (i)–(ii), (iii)–(iv) and (v)–(vi), respectively. We can not comment on the value function for parameter values outside the shaded regions.

The next result characterizes the number and the level of impulses in the FNE.

Theorem 2.7 *Let Assumption 2.1 and 2.4 hold. Let $As_2 + w_2 \neq 0$ when $s_2 = \gamma$ or $s_2 = -\gamma$. There can exist at most two impulses in the FNE, that is, $k^* \leq 2$.*

(i) If the parameters satisfy either of the following conditions, then an impulse occurs at $\tau^1_{fb} = 0$:

(a) with $A = 0$: either $Tw_2 + s_2 = \gamma$ or $Tw_2 + s_2 = -\gamma$,

(b) with $A \neq 0$: either $As_2e^{AT} + w_2(e^{AT} - 1) - A\gamma = 0$ or $As_2e^{AT} + w_2(e^{AT} - 1) + A\gamma = 0$.

(ii) If either $s_2 = \gamma$, or $s_2 = -\gamma$, then an impulse occurs at $\tau_{fb}^2 = T$.

(iii) If the parameters satisfy either of the following conditions, then there are exactly two impulses at $\tau_{fb}^1 = 0$ and $\tau_{fb}^2 = T$:

$$A = 0,\ s_2 = -\gamma,\ Tw_2 = 2\gamma, \tag{2.50a}$$

$$A = 0,\ s_2 = \gamma,\ Tw_2 = -2\gamma, \tag{2.50b}$$

$$A \neq 0,\ s_2 = -\gamma,\ w_2 = A\gamma\frac{e^{AT}+1}{e^{AT}-1}, \tag{2.50c}$$

$$A \neq 0,\ s_2 = \gamma,\ w_2 = -A\gamma\frac{e^{AT}+1}{e^{AT}-1}. \tag{2.50d}$$

The equilibrium control of Player 1 when $k^* = 1$, impulse occurs at initial time and $t \in (0, T]$ is

$$u(t) = \begin{cases} \frac{B}{R_1}\left(\frac{w_1}{A} - (s_1 + \frac{w_1}{A})e^{A(T-t)}\right), & A \neq 0 \\ \frac{B}{R_1}\left(w_1(t-T) - s_1\right), & A = 0 \end{cases}, \tag{2.51a}$$

and when impulse occurs at the final time and $t \in [0, T)$, we have

$$u(t) = \begin{cases} \frac{B}{R_1}\left(\frac{w_1}{A} - (s_1 + q_1 + \frac{w_1}{A})e^{A(T-t)}\right), & A \neq 0 \\ \frac{B}{R_1}\left(w_1(t-T) - (s_1 + q_1)\right), & A = 0 \end{cases}. \tag{2.51b}$$

If either $A = 0$ and $w_2 < 0$, or $A \neq 0$ and $As_2 + w_2 < 0$, then, for $k^* = 1$, the equilibrium impulse levels of Player 2 for impulses at $\tau_{fb}^1 = 0$, $\tau_{fb}^2 = T$ are given by

$$v_{fb}^1 = -\operatorname{sign}(Q)\sqrt{\frac{2C}{P_2}},\ v_{fb}^2 = \operatorname{sign}(Q)\sqrt{\frac{2C}{P_2}}. \tag{2.52a}$$

If either $A = 0$ and $w_2 > 0$, or $A \neq 0$ and $As_2 + w_2 > 0$, then, for $k^* = 1$, the equilibrium impulse levels of Player 2 for impulses at $\tau_{fb}^1 = 0$, $\tau_{fb}^2 = T$ are given by

$$v_{fb}^1 = \operatorname{sign}(Q)\sqrt{\frac{2C}{P_2}},\ v_{fb}^2 = -\operatorname{sign}(Q)\sqrt{\frac{2C}{P_2}}. \tag{2.53a}$$

Proof. In Theorem 2.5, it is shown that impulses can occur at $\tau_{\text{fb}}^1 = 0$ and $\tau_{\text{fb}}^2 = T$ only. We know from (2.43a) and (2.43b) that

$$\alpha_2(t) = \begin{cases} w_2(T-t) + s_2, & A = 0, \\ -\frac{w_2}{A} + e^{A(T-t)}\left(s_2 + \frac{w_2}{A}\right), & A \neq 0. \end{cases} \tag{2.54a}$$

From (2.48), an impulse occurs when $\alpha_2(t)^2 = \gamma^2$. For an impulse to occur at $\tau^1_{\text{fb}} = 0$, we have either $\alpha_2(0) = \gamma$ or $\alpha_2(0) = -\gamma$. Similarly, an impulse occurs at $\tau^2_{\text{fb}} = T$ when $\alpha_2(T) = \gamma$ or $\alpha_2(T) = -\gamma$.

Also, $\alpha_2(t)$ is strictly monotone in time. So, two impulses occur at initial and final time if either $\alpha_2(0) = -\gamma$ and $\alpha_2(T) = \gamma$ or $\alpha_2(T) = \gamma$ and $\alpha_2(0) = \gamma$, that is, conditions (2.50a) –(2.50d) hold true.

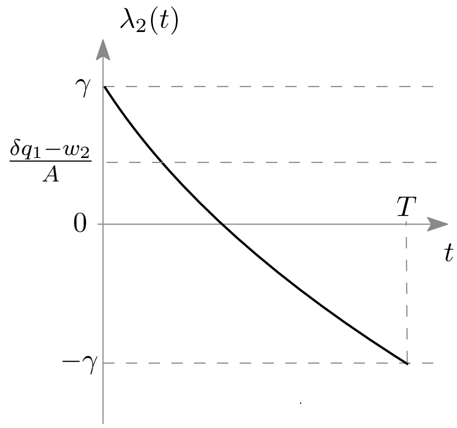
Next, we characterize the FNE of the differential game (2.2)–(2.5) when impulses occur at $\tau^1_{\text{fb}} = 0$ and $\tau^2_{\text{fb}} = T$. The equilibrium controls of Player 1 given in (2.51) are obtained by first solving for $\alpha_1(.)$ from (2.37a) and (2.38a), and then using $u(t) = -\frac{B}{R_1}\alpha_1(t)$. To obtain the equilibrium impulse levels for Player 2, we insert $\alpha_2(t)$ evaluated at $t = 0$ and $t = T$ from (2.43a) and (2.43b) in (2.45). The impulse levels are given by $v^1_{\text{fb}} = -\frac{Q}{P_2}\alpha_2(0)$ and $v^2_{\text{fb}} = -\frac{Q}{P_2}\alpha_2(T)$. When $\alpha_2(t)$ is increasing (decreasing) in time, we have $\alpha_2(0) = -\gamma$ ($\alpha_2(0) = \gamma$) and $\alpha(T) = \gamma$ ($\alpha_2(T) = -\gamma$). Therefore, the impulse levels are given by (2.52), (2.53) depending on the problem parameters. ■

Remark 2.8 *We have the following observations: (i) The level of impulse is a constant and proportional to the ratio of fixed cost C and the coefficient of proportional transaction cost P_2. Note that P_2 can be interpreted as the marginal cost at zero impulse, i.e.,* $\left.\frac{\partial\left(\frac{1}{2}P_2 v_i^2\right)}{\partial v_i}\right|_{v_i=0}$*. (ii) The timing of an impulse by Player 2 is independent of Player 1's parameters. Indeed, it depends on Player 2's parameter values and the coefficient entering the state dynamics. Finally, (iii) when there are two impulses, the magnitude of the impulses is the same and they are opposite in sign.*

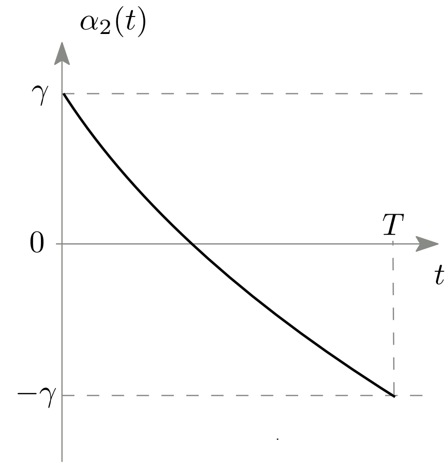
2.4.3 Comparison of open-loop and feedback Nash equilibria

From Theorems 2.4 and 2.7, it is clear that OLNE and FNE do not coincide when the number and timing of impulse instants are decision variables of Player 2. In the following, we highlight reasons as to why these equilibria differ in the endogenous case.

In the OLNE, the Hamiltonian continuity condition (2.20f) reduces to an affine function of $\lambda_2(t)$ in $(0, T)$ whereas at $t = 0$ and $t = T$, we obtain an inequality that is quadratic in $\lambda_2(t)$. Since the co-state is strictly monotone, at most three impulses can occur, see Figure 2.3a.



(a) Variation of $\lambda_2(t)$



(b) Variation of $\alpha_2(t)$

Figure 2.3 – Variation of $\lambda_2(t)$ and $\alpha_2(t)$ for three impulses in OLNE whereas there are two impulses in FNE

In the FNE, the continuation set is characterized by the time interval during which the gradient of the value function of Player 2 satisfies $-\gamma < \alpha_2(t) < \gamma$. The stopping set is characterized by the time instants at which $\alpha_2(t)$ takes a value of either γ or $-\gamma$. There is no dependence of stopping set on the equilibrium control of Player 1 while in the OLNE, the Hamiltonian continuity condition, which determines the impulse timing, depends on the equilibrium control of Player 1. From (2.43), $\alpha_2(t)$ is strictly monotone function of time, and it can achieve a maximum and minimum value of γ and -γ at $t = 0$ or $t = T$ for all $x \in \mathbb{R}$; see Figure 2.3b.

Remark 2.9 *When both the continuous payoff and salvage value of Player 2 either increase in x or decrease in x, i.e., if $w_2 > 0, s_2 > 0$ or $w_2 < 0, s_2 < 0$, then it is clear from Figure 2.2 that there can be at most one impulse in the FNE while from Figure 2.1, there can be at most three impulses in the OLNE.*

Now, we study the open-loop and feedback Nash equilibrium solutions for the parameter regions where the value function of Player 2 is well-defined.

(i) Assume that Player 2 incurs a running cost, i.e., $w_2 < 0$ and that the salvage value of Player 2 is decreasing in x, i.e., $s_2 < 0$. Also, assume that $w_2 \neq q_1\delta$ when $A = 0$, and $As_2 + w_2 \neq 0$ when $s_2 = \gamma$ or $s_2 = -\gamma$.

With $A = 0$, an impulse can occur at the initial time in the OLNE and FNE when $Tw_2 + s_2 = -\gamma$. However, for other parameter values in the shaded region in Figure 2.2a, there are no impulses in the FNE, while an impulse can occur at the initial time in the OLNE for all $w_2 < 0, s_2 < 0$; see Figure 2.1a, Figure 2.1b.

With $A > 0$, τ_{ol}^I is the interior impulse in the OLNE if $q_1 < 0$ and $q_1\delta < As_2 + w_2 < q_1\delta e^{-AT}$ (see Figure 2.1d). Further, there can be at most three impulses in the OLNE when $q_1 < 0$. FNE has no interior impulses and $\tau_{\text{fb}}^1 = 0$ is an impulse instant when $As_2e^{AT} + w_2(e^{AT} - 1) + A\gamma = 0$ and $\tau_{\text{fb}}^2 = T$ is an impulse instant for $s_2 = -\gamma$. For the other parameter values in the shaded region in Figure 2.2b, there is no impulse in the FNE.

With $A < 0$, τ_{ol}^I is the interior impulse in the OLNE if $q_1 > 0$ and $q_1\delta < As_2 + w_2 < q_1\delta e^{-AT}$ (see Figure 2.1e), or $q_1 < 0$ and $q_1\delta e^{-AT} < As_2 + w_2 < q_1\delta$ (see Figure 2.1f). In the OLNE, there can be at most three impulses when $q_1 > 0$. In the FNE, an impulse occurs at $\tau_{\text{fb}}^1 = 0$ when $As_2e^{AT} + w_2(e^{AT} - 1) + A\gamma = 0$, $\tau_{\text{fb}}^2 = T$ is an impulse instant when $s_2 = -\gamma$, and for other parameter values, there is no impulse; see Figure 2.2c.

(ii) Second, we assume that Player 2 values the state positively so that $w_2 > 0$, and her salvage value is increasing in x, i.e., $s_2 > 0$. Also, assume that $w_2 \neq q_1\delta$ when $A = 0$, and $As_2 + w_2 \neq 0$ when $s_2 = \gamma$ or $s_2 = -\gamma$.

With $A = 0$, an impulse can occur at the initial time in the OLNE and FNE when $Tw_2 + s_2 = \gamma$; see Figure 2.1a, 2.1b, 2.2a. There are no impulses in the FNE for any other parameter value while an impulse can occur in the OLNE for all $w_2 > 0, s_2 > 0$.

With $A > 0$, τ_{ol}^I is the interior impulse in the OLNE if $q_1 > 0$ and $q_1\delta e^{-AT} < As_2 + w_2 < q_1\delta$ (see Figure 2.1c). There can be at most three impulses in the OLNE when $q_1 > 0$. In the FNE, $\tau_{\text{fb}}^1 = 0$ is an equilibrium impulse instant when $As_2e^{AT} + w_2(e^{AT} - 1) - A\gamma = 0$ and $\tau_{\text{fb}}^2 = T$ is an equilibrium impulse instant when $s_2 = \gamma$. For other parameter values, there is no impulse in the FNE. (see Figure 2.2b).

With $A < 0$, τ_{ol}^I is the interior impulse in the OLNE if $q_1 > 0$ and $q_1\delta < As_2 + w_2 < q_1\delta e^{-AT}$ (see Figure 2.1e) or $q_1 < 0$ and $q_1\delta e^{-AT} < As_2 + w_2 < q_1\delta$ (see Figure 2.1f). There can be at most three impulses in the OLNE if $q_1 < 0$.

In the FNE, $\tau_{\text{fb}}^1 = 0$ is an impulse instant when $As_2e^{AT} + w_2(e^{AT} - 1) - A\gamma = 0$, $\tau_{\text{fb}}^2 = T$ is an impulse instant when $s_2 = \gamma$, and for other parameter values, there is no impulse; see Figure 2.2c.

2.5 Numerical example

In this section, we illustrate our results with a numerical example.

In the literature, a linear-state differential game problem (see Novak et al., 2010), Crettez and Hayek (2014) between a government and an international terrorist organization (ITO) has been studied where government's utility is linearly decreasing with ITO's resources while ITO's utility is increasing linearly with its own resources. As a result, the government launches strikes to disrupt the infrastructure (resources) of the ITO. Motivated by this example and recent research on attacker-defender dynamic game models (Etesami and Başar, 2019), we consider a two-player differential game between Player 1 who values the state positively and Player 2 who values the state negatively. For instance, Player 1 can be a firm that aims to increase the security level of a system and invests effort in reducing the system vulnerabilities while Player 2 is an attacker that invests effort in reducing the security level of a system. Player 2 uses an impulse control that consists of determining the number $k \in \mathbb{N}$ and timing of impulses τ_i, $(i = 1, 2, \cdots, k)$ in addition to the corresponding effort level v_i. We consider that at the impulse times, Player 2 incurs a fixed cost, and a variable cost that is quadratic in the effort level v_i. The fixed cost discourages Player 2 to intervene frequently. The security level, which denotes the state of the system, evolves according to the following dynamics and the jump equation:

$$\begin{aligned} &\dot{x}(t) = -0.1x(t) + 0.6u(t),\ x(0^-) = 5, \\ &x(\tau_i^+) - x(\tau_i^-) = 0.2v_i, \end{aligned}$$

The objective functions of Player 1 and 2 are given by

$$\begin{aligned} J_1 &= \int_0^T [4x(t) - 0.5u(t)^2]dt - \sum_{i=1}^{k} 0.3x(\tau_i^-) + x(T^+), \\ J_2 &= -\int_0^T 0.8x(t)dt - \sum_{i=1}^{k}(0.1v_i^2 + 1) - x(T^+), \end{aligned}$$

where $T = 5$.

Under the open-loop information structure and using the necessary conditions, the candidate solution for impulse in OLNE is $\tau = 2.4$. The OLNE is given by $(u^*_{\text{ol}}(.),$
$(2.4, -2.6), k^* = 1)$ where equilibrium effort for Player 1 is given by

$$u^*_{\text{ol}}(t) = \begin{cases} 24 - 14.33\,\mathrm{e}^{0.1t} & t \in [0, 2.4), \\ 24 - 14.19\,\mathrm{e}^{0.1t} & t \in (2.4, T]. \end{cases}$$

The open-loop Nash equilibrium payoff of Player 1 is 167.98 while Player 2 obtains a payoff of -66.42. The FNE is given by $(u^*_{\text{fb}}(t), k^* = 0\})$ where Player 2 does not give any impulse, and the equilibrium effort of Player 1 is given by

$$u^*_{\text{fb}}(t) = 24 - 14.19\,\mathrm{e}^{0.1t},\ t \in [0, T].$$

The equilibrium payoff of Player 1 is given by 177.31, and Player 2 obtains a payoff of -66.83.
Next, we consider the following objective for Player 2:

$$J_2 = -\int_0^T 0.34x(t)dt - \sum_{i=1}^{k}(0.1v_i^2 + 1) - 3x(T^+),$$

while keeping the other parameter values as before. In this case, the candidate open-loop Nash equilibrium strategy of Player 2 is to give an impulse at the final time T. The OLNE and FNE are given by $(u^*_{\text{ol}}, (T, -3), k^* = 1)$ and $(u^*_{\text{fb}}, (0, -3.16), k^* = 1)$ where

$$u^*_{\text{ol}}(t) = 24 - 14.30\,\mathrm{e}^{0.1t},\ t \in [0, T),$$
$$u^*_{\text{fb}}(t) = 24 - 14.19\,\mathrm{e}^{0.1t},\ t \in (0, T].$$

The equilibrium payoff of Player 1 and Player 2 in OLNE is 172.17 and -66.71, respectively while in the FNE, Player 1 and 2 obtain 165.47 and -67.93, respectively.

In both cases, we see that Player 1 uses controls that increases the state while Player 2's equilibrium impulse decreases the state value. When compared with the OLNE, Player 2 obtains a lower payoff in the FNE. Due to the state-dependent costs incurred because of the intervention by Player 2 in $(0, T]$, Player 1's equilibrium strategy is to invest lower effort in OLNE when compared with the FNE.

2.6 Some extensions

In this section, we consider two extensions of the canonical differential game model described by (2.2-2.5). In particular, we show that the conclusions obtained in Sections 2.3 and 2.4 remain unaltered, qualitatively, for the following extensions.

2.6.1 General cost structures

Suppose the piecewise continuous control of Player 1 involves a cost $d(u)$ and the variable cost of impulse for Player 2 is given by $c(v_i)$. We make the following assumption to obtain a unique expression for piecewise continuous control of Player 1 and for the impulse level of Player 2.

Assumption 2.5 *We assume that the functions $d : \Omega_u \to \mathbb{R}$ and $c : \Omega_v \to \mathbb{R}$ are continuous and twice continuously differentiable. Further, we assume that these functions admit interior maxima, and satisfy $\frac{\partial^2[d(u)]}{\partial u^2} < 0$ over Ω_u and $\frac{\partial^2[c(v)]}{\partial v^2} < 0$ over Ω_v.*

Theorem 2.8 *Let Assumption 2.1 and 2.5 hold, and assume that the value functions of both players are linear in state. Then the open-loop and feedback Nash equilibria of the differential game (2.2–2.5) coincide when the number and timing of impulses is exogenously given. When the number and timing of impulse instants are decision variables of Player 2, then these two equilibria are different.*

Proof. See Appendix 2.8.2. ■

In the above theorem, we showed that our results hold qualitatively when we consider a general cost structure. Next, we analyze the multi-dimensional extension of our scalar LSDG model.

2.6.2 Multi-dimensional state

We consider a multi-dimensional extension of the linear-state game described by (2.2–2.5), and examine if the conclusions derived in Sections 2.3 and 2.4 still hold true. Towards this end, we assume that the state variable is an n-dimensional vector, and the controls satisfy $u(t) \in \mathbb{R}^{m_1}$ and $v \in \mathbb{R}^{m_2}$. The parameters in (2.2–2.5) are $w_1, w_2, q_1, s_1, s_2 \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $B \in \mathbb{R}^{n\times m_1}$, $Q \in \mathbb{R}^{n\times m_2}$, $R_1 \in \mathbb{R}^{m_1\times m_1}$, $P_2 \in \mathbb{R}^{m_2\times m_2}$.

With exogenously given impulse instants, we use the necessary conditions (2.7a)–(2.7e), (2.9a)–(2.9e) to obtain the equilibrium control $u^*(t)$ of Player 1 and equilibrium impulse level v_i^* of Player 2. Under the feedback information structure, we can use the dynamic programming principle to show that the gradients of the value functions of Player 1 and Player 2 given in Assumption 2.2 are equal to the co-states of players in the OLNE, and the equilibrium controls are the same for both the players which implies that OLNE and FNE coincide. When the impulse instants are decision variables of Player 2, the equilibrium impulse instants satisfy (2.20f) where the difference of the Hamiltonian of Player 2 before and after the equilibrium impulse instant is given by

$$\left(q_1' B R_1^{-1} B' - (w_2' + \lambda_2(\tau_i^*)' A) Q P_2^{-1} Q'\right) \lambda_2(\tau_i^*), \tag{2.55}$$

where $\lambda_2(t) \in \mathbb{R}^n$. In the FNE, value function of the players given in Assumption 2.4 satisfy (2.32b) with equality. Therefore, the stopping set $\mathcal{S}$ and the continuation set $\mathcal{C}$ are given by

$$\begin{aligned} \mathcal{S} &= \{(t,x) |\, \alpha_2(t)' Q P_2^{-1} Q' \alpha_2(t) = 2C\}, \\ \mathcal{C} &= \{(t,x) |\, \alpha_2(t)' Q P_2^{-1} Q' \alpha_2(t) > 2C\}. \end{aligned}$$

where $\alpha_2(t) \in \mathbb{R}^n$. Similar to the scalar case, we have that both the stopping set and continuation set are independent of the state of the system. Also, the impulse timing is completely determined by the problem parameters of Player 2 only whereas it is clear from (2.55) that the impulse instants in OLNE depend on the problem parameters of Player 1.

2.7 Conclusions

In this paper, we determined open-loop and feedback Nash equilibria in the scalar deterministic finite-horizon two-player nonzero-sum linear-state differential game with impulse controls, in two cases, namely, when the impulse instants are given and when Player 2 endogenously determines the equilibrium timing of the impulses. We showed that open-loop and feedback equilibria coincide when the impulse instants are exogenously given, and that they differ when these instants are endogenously determined.

For future research, it would be interesting to determine the feedback solutions for more general classes of differential games with impulse controls. A natural first candidate is the class of linear-quadratic differential games, which is often used in applications. Clearly, there would be computational challenges since the stopping set condition would involve the state variables that evolve forward in time, while the Ricatti system of Player 1 and Player 2 evolve backwards in time. Another extension of this work could be to consider the case where both players use piecewise continuous as well as impulse controls.

2.8 Appendix

2.8.1 Proof of Theorem 2.2

Assuming that the equilibrium strategy of Player 2 is given by $\tilde{v}^*$, Player 1 solves (2.15). The Hamilton-Jacobi-Bellman (HJB) equation for Player 1 for $t \in (\tau_i^+, \tau_{i+1}^-)$, $i \in \{0, 1, \cdots, k\}$ is given by

$$-\frac{\partial V_1(t,x)}{\partial t} = \max_{u \in \Omega_u} \big(w_1 x + \frac{1}{2} R_1 u(t)^2 + \left(\frac{\partial V_1}{\partial x}\right)(Ax + Bu(t))\big).$$

Under Assumption 2.2, we can rewrite the HJB equation as

$$-\dot{m}_1(t)x - \dot{n}_1(t) = \max_{u \in \Omega_u} \big(w_1 x + \frac{1}{2} R_1 u(t)^2 + m_1(t)(Ax + Bu(t))\big).$$

Since we have assumed that the equilibrium controls lie in the interior of Ω_u (see Assumption 2.1), the first-order condition gives:

$$u^*(t) = -\frac{B}{R_1}\frac{\partial V_1(t,x)}{\partial x} = -\frac{Bm_1(t)}{R_1}. \tag{2.56}$$

Using the equilibrium control in the HJB equation, we obtain

$$-\dot{m}_1(t)x - \dot{n}_1(t) = w_1 x - \frac{B^2 m_1(t)^2}{2R_1} + Am_1(t)x.$$

On comparing the coefficients, we have

$$\dot{m}_1(t) = -w_1 - Am_1(t),\ m_1(T^+) = s_1, \tag{2.57a}$$

$$\dot{n}_1(t) = \frac{B^2 m_1(t)^2}{2R_1},\ n_1(T^+) = 0. \tag{2.57b}$$

At the impulse instants, the value functions are related as follows:

$$V_1(\tau_i^-, x(\tau_i^-)) = V_1(\tau_i^+, x(\tau_i^-) + Qv_i) + q_1 x(\tau_i^-), \tag{2.58}$$

where v_i^* denotes the equilibrium impulse level used by Player 2 at the impulse instant τ_i. Using $V_1(t,x) = m_1(t)x + n_1(t)$, we obtain

$$m_1(\tau_i^-)x(\tau_i^-) + n_1(\tau_i^-) = m_1(\tau_i^+)x(\tau_i^-) + m_1(\tau_i^+)Qv_i + n_1(\tau_i^+) + q_1 x(\tau_i^-),$$

which results in the following update equations for $m_1(.)$ and $n_1(.)$:

$$m_1(\tau_i^-) = m_1(\tau_i^+) + q_1, \tag{2.59a}$$
$$n_1(\tau_i^-) = n_1(\tau_i^+) + m_1(\tau_i^+)Qv_i. \tag{2.59b}$$

Given the equilibrium strategy $u^*(.)$ of Player 1, Player 2 solves (2.16). For the impulse-free region, we have the following relation:

$$-\frac{\partial V_2(t,x)}{\partial t} = w_2 x + \left(\frac{\partial V_2}{\partial x}\right)(Ax + Bu^*(t)),$$

which, on substituting the equilibrium control $u^*(t)$ of Player 1 and the value function of Player 2, $V_2(t,x) = m_2(t)x + n_2(t)$ (see Assumption 2.2) simplifies to

$$w_2 x + \dot{m}_2(t)x + \dot{n}_2(t) + m_2(t)(Ax - \frac{B^2 m_1(t)}{R_1}) = 0.$$

On comparing the above coefficients, we get for $t \neq \{\tau_1, \tau_2, \cdots, \tau_k\}$,

$$\dot{m}_2(t) = -w_2 - Am_2(t),\ m_2(T^+) = s_2, \tag{2.60a}$$
$$\dot{n}_2(t) = \frac{B^2 m_1(t) m_2(t)}{R_1},\ n_2(T^+) = 0. \tag{2.60b}$$

At the impulse instants $\{\tau_2, \tau_2, \cdots, \tau_k\}$, the equilibrium value function of Player 2 satisfies

$$V_2(\tau_i^-, x(\tau_i^-)) = \max_{v_i \in \Omega_v} \{V_2(\tau_i^+, x(\tau_i^-) + Qv_i) + \frac{1}{2}P_2 v_i^2 + C\}. \tag{2.61}$$

The above equation implies that, at the impulse instant, Player 2 selects the equilibrium control to maximize the value-to-go from that instant onwards. From Assumption 2.1 on interior solution, the equilibrium impulse level is obtained as follows:

$$v_i^* = \arg\max_{v_i \in \Omega_v} \left\{ V_2(\tau_i^+, x(\tau_i^-) + Qv_i) + \frac{1}{2}P_2 v_i^2 + C \right\}$$

$$= \arg\max_{v_i \in \Omega_v} \left\{ m_2(\tau_i^+)(x(\tau_i^-) + Qv_i) + n_2(\tau_i^+) + \frac{1}{2}P_2 v_i^2 + C \right\}$$

$$= -\frac{m_2(\tau_i^+)Q}{P_2}. \tag{2.62a}$$

Using v_i^* in (2.61), we obtain

$$m_2(\tau_i^-)x(\tau_i^-) + n_2(\tau_i^-) = m_2(\tau_i^+)x(\tau_i^-) + n_2(\tau_i^+) - \frac{m_2(\tau_i^+)^2 Q^2}{2P_2} + C.$$

The above relation holds for all x. Therefore, we have

$$m_2(\tau_i^-) = m_2(\tau_i^+), \tag{2.63a}$$

$$n_2(\tau_i^-) = n_2(\tau_i^+) - \frac{m_2(\tau_i^+)^2 Q^2}{2P_2} + C. \tag{2.63b}$$

Using (2.57a), (2.59a), we obtain that for $A \neq 0$

$$m_1(t) = -\frac{w_1}{A} + \left(m_1(\tau_{j+1}^-) + \frac{w_1}{A} \right) e^{A(\tau_{j+1}^- - t)}, \ \forall t \in (\tau_j, \tau_{j+1}), j \in \{0, 1, \cdots, k\},$$

where $m_1(\tau_{k+1}^+) = s_1$, and

$$m_1(\tau_i^-) = -\frac{w_1}{A} + \left(m_1(\tau_{i+1}^-) + \frac{w_1}{A} \right) e^{A(\tau_{i+1}^- - \tau_i^+)} + q_1,$$

for $i \in \{1, 2, \cdots, k\}$. For $A = 0$, we obtain

$$m_1(t) = w_1(\tau_{j+1}^- - t) + m_1(\tau_{j+1}^-), \ \forall t \in (\tau_j, \tau_{j+1}), \ j \in \{0, 1, \cdots, k\},$$

where $m_1(\tau_{k+1}^+) = s_1$, and

$$m_1(\tau_i^-) = w_1(\tau_{i+1}^- - \tau_i^+) + m_1(\tau_{i+1}^-) + q_1,$$

where $i \in \{1, 2, \cdots, k\}$. From (2.60a), (2.63a), we obtain that for $A \neq 0$,

$$m_2(t) = -\frac{w_2}{A} + \left(s_2 + \frac{w_1}{A} \right) e^{A(T-t)}, \forall t \in [0, T],$$

and for $A = 0$,

$$m_2(t) = w_2(T - t) + s_2, \forall t \in [0, T].$$

From (2.56) and (2.62a), the equilibrium controls are given in (2.18a) and (2.18b) for $A \neq 0$ and in (2.19a) and (2.19b) for $A = 0$.

2.8.2 Proof of Theorem 2.8

First, we consider the case when the impulse instants are exogenously given. The OLNE strategies $u^*(t)$ and v_i^* of the players are obtained by solving (2.7a) and (2.9c) where the Hamiltonian of Player 1 and Player 2, and the impulse Hamiltonian of Player 2 are respectively given by

$$H_1(x(t), u(t), \lambda_1(t)) = w_1 x(t) + d(u(t)) + \lambda_1(t)(Ax(t) + Bu(t)), \quad \text{(2.64a)}$$

$$H_2(x(t), u(t), \lambda_2(t)) = w_2 x(t) + \lambda_2(t)(Ax(t) + Bu(t)), \quad \text{(2.64b)}$$

$$H_2^I(x(t), v_i, \lambda_2(t)) = C + c(v_i) + \lambda_2(t) Q v_i. \quad \text{(2.64c)}$$

From Assumption 2.1, the first-order conditions in (2.7a) and (2.9c) give

$$H_{1u}(x(t), u^*(t), \lambda_1(t) = 0 \Rightarrow d_u(u^*(t)) + B\lambda_1(t) = 0,$$

$$H_{2v_i}^I(x(\tau_i^-), v_i^*, \lambda_2(\tau_i^+)) = 0 \Rightarrow c_{v_i}(v_i^*) + \lambda_2(\tau_i^+) Q = 0.$$

Following Assumption 2.5, and from implicit function theorem, there exist continuously differentiable functions $f_1 : \mathbb{R} \to \Omega_u$ and $f_2 : \mathbb{R} \to \Omega_v$ such that

$$u^*(t) = f_1(B\lambda_1(t)), \quad \text{(2.66a)}$$

$$v_i^* = f_2(Q\lambda_2(\tau_i^+)). \quad \text{(2.66b)}$$

From (2.7b)–(2.7e) and (2.9a)–(2.9e), it follows that $\lambda_1(t)$ and $\lambda_2(t)$ satisfy (2.12c)–(2.12d), (2.12g)–(2.12h), and the state equations for $i \in \{1, 2, \cdots, k\}$ are given by

$$\dot{x}(t) = Ax(t) + Bf_1(B\lambda_1(t)), \text{ for } t \neq \tau_i, \; x(0^-) = x_0, \quad \text{(2.67a)}$$

$$x(\tau_i^+) = x(\tau_i^-) + Qf_2(Q\lambda_2(\tau_i^+)). \quad \text{(2.67b)}$$

Next, we consider the feedback information structure, and use the dynamic programming principle to obtain the FNE strategies of the players. Since we have considered a linear-state differential game, we assume that the value functions of Player 1 and Player 2 are given by (2.17a) and (2.17b). Between the impulse instants, the value function of Player 1 satisfies the HJB equation

$$-\dot{m}_1(t)x - \dot{n}_1(t) = \max_{u \in \Omega_u} \big(w_1 x + d(u(t)) + m_1(t)(Ax(t) + Bu(t))\big).$$

Following Assumption 2.1, the first-order condition yields $d_u(u^*(t))+Bm_1(t)=0$. From Assumption 2.5 and from implicit function theorem, there exist continuously differentiable functions $f_1 : \mathbb{R} \to \Omega_u$ such that

$$u^*(t) = f_1(Bm_1(t)). \tag{2.68}$$

For optimal control $u^*(t)$, the HJB equation is then given by

$$-\dot{m}_1(t)x - \dot{n}_1(t) = w_1 x + d(u^*(t)) + m_1(t)(Ax(t) + Bu^*(t)).$$

On comparing the coefficients, we obtain

$$\dot{m}_1(t) = -w_1 - m_1(t)A, \tag{2.69a}$$

$$\dot{n}_1(t) = -d(u^*(t)) - m_1(t)Bu^*(t). \tag{2.69b}$$

The jump in the value function of Player 1 is given by (2.58)

$$\begin{aligned} m_1(\tau_i^-)x(\tau_i^-) + n_1(\tau_i^-) =& m_1(\tau_i^+)x(\tau_i^+) + n_1(\tau_i^+) + q_1 x(\tau_i^-) \\ &= m_1(\tau_i^+)(x(\tau_i^-) + Qv_i^*) + n_1(\tau_i^+) + q_1 x(\tau_i^-). \end{aligned}$$

On comparing the coefficients, we obtain

$$m_1(\tau_i^-) = m_1(\tau_i^+) + q_1, \tag{2.70}$$

$$n_1(\tau_i^-) = n_1(\tau_i^+) + m_1(\tau_i^+)Qv_i^*. \tag{2.71}$$

Between the impulse instants, the value function of Player 2 (2.17b) satisfies the HJB equation given by

$$-\dot{m}_2(t)x - \dot{n}_2(t) = w_2 x + m_2(t)(Ax(t) + Bu^*(t)).$$

On comparing the coefficients, we obtain

$$\dot{m}_2(t) = -w_2 - m_2(t)A, \tag{2.72a}$$

$$\dot{n}_2(t) = -m_2(t)Bu^*(t). \tag{2.72b}$$

At the impulse instant τ_i, the value function of Player 2 satisfies

$$m_2(\tau_i^-)x(\tau_i^-) + n_2(\tau_i^-) = \max_{v_i \in \Omega_v} \big(m_2(\tau_i^+)(x(\tau_i^-) + Qv_i) + n_2(\tau_i^+) + C + c(v_i)\big). \tag{2.73}$$

From Assumption 2.1, the first-order condition yields

$$m_2(\tau_i^+)Q + c_{v_i}(v_i^*) = 0.$$

Following Assumption 2.5, and from the implicit function theorem, there exist continuously differentiable functions $f_2 : \mathbb{R} \to \Omega_v$ such that

$$v_i^* = f_2(Qm_2(\tau_i^+)). \tag{2.74}$$

On substituting v_i^* in (2.73), we obtain

$$m_2(\tau_i^-)x(\tau_i^-) + n_2(\tau_i^-) = m_2(\tau_i^+)(x(\tau_i^-) + Qv_i^*) + n_2(\tau_i^+) + C + c(v_i^*),$$

which on comparing coefficients gives

$$m_2(\tau_i^-) = m_2(\tau_i^+), \tag{2.75}$$

$$n_2(\tau_i^-) = n_2(\tau_i^+) + m_2(\tau_i^+)Qv_i^* + C + c(v_i^*). \tag{2.76}$$

The necessary conditions for OLNE require that co-state variables of Player 1 and Player 2 satisfy (2.12c)-(2.12d), (2.12g)–(2.12h). For the FNE, the gradient of the value function of Player 1 and Player 2 are obtained by solving (2.69a), (2.70), (2.72a), and (2.75). For both players, $\lambda_1(t) = m_1(t)$, $\lambda_2(t) = m_2(t)$ for all t since $\lambda_1(.)$ and $m_1(.)$, and $\lambda_2(.)$ and $m_2(.)$ have the same dynamics, jump conditions, and terminal conditions. Therefore from (2.66), (2.68), (2.74), we have that OLNE and FNE coincide when the impulse timing is given.

When the impulse instants are decision variables of Player 2, the necessary conditions for OLNE are given in (2.7) and (2.20). Using the necessary conditions, and from Assumption 2.1 on interior solutions, the equilibrium controls are given in (2.66a)–(2.66b) where the dynamics and jump equations of co-state variables are given by (2.12c)–(2.12d), (2.12g)–(2.12h). For an impulse to occur in $[0, T]$, (2.20f) must hold true which on substituting (2.64b), (2.66a), (2.12h), (2.67b) simplifies to

$$(w_2 + A\lambda_2(\tau_i^*))Qf_2(Q\lambda_2(\tau_i^*)) + \lambda_2(\tau_i^*)B(f_1(B\lambda_1(\tau_i^{*+})) - f_1(B\lambda_1(\tau_i^{*-})))$$
$$\begin{cases} > 0 & \text{for } \tau_i^* = 0 \\ = 0 & \text{for } \tau_i^* \in (0, T) \\ < 0 & \text{for } \tau_i^* = T \end{cases}.$$

From the above condition, it is clear that the equilibrium impulse instant in OLNE depends on the problem parameters of Player 1.

Next, we consider the feedback information structure. Given Player 1's equilibrium strategy $u^*(.)$, Player 2 solves (2.29). We assume linear value function for both players, that is,

$$V_i(t, x) = \alpha_i(t)x + \beta_i(t),\ \forall i = \{1, 2\}.$$

Since Player 2 solves an impulse optimal control problem, the value function of Player 2 satisfies the QVI (2.32). The stopping set is characterized by the time instant at which (2.32b) holds with equality, that is,

$$V_2(t, x) = \mathcal{R}V_2(t, x) \Rightarrow \alpha_2(t, x) + \beta_2(t) = \max_{v_i \in \Omega_v}\{\alpha_2(t)(x + Qv_i) + \beta_2(t) + C + c(v_i)\}.$$

From Assumption 2.1 on interior solutions, the first-order condition gives $\alpha_2(t)Q + c_{v_i}(v_i^*) = 0$. From Assumption 2.5, we can write

$$v_i^* = f_2(Q\alpha_2(t)). \tag{2.77}$$

For the equilibrium control v_i^*, we obtain the stopping set condition

$$\alpha_2(t)Qf_2(Q\alpha_2(t)) + c(f_2(Q\alpha_2(t))) + C = 0. \tag{2.78}$$

Since (2.32b) must hold for all $(t, x) \in [0, T] \times \mathbb{R}$, the linear value function is well-defined when the following condition holds for all $t \in [0, T]$.

$$\alpha_2(t)Qf_2(Q\alpha_2(t)) + c(f_2(Q\alpha_2(t))) + C \leq 0.$$

Following the proof of Theorem 2.5, we obtain the gradient of the value function of Player 2 as follows:

$$\alpha_2(t) = w_2(T - t) + s_2,\ A = 0,$$

$$\alpha_2(t) = -\frac{w_2}{A} + e^{A(T-t)}\left(s_2 + \frac{w_2}{A}\right),\ A \neq 0.$$

The stopping set condition (2.78) implies that the impulse timing only depends on the problem parameters of Player 2, and is independent of the state of the system. On the other hand, the impulse timing in OLNE involves problem parameters of Player 1. Therefore, OLNE and FNE do not coincide when Player 2 decides the number and timing of impulses.

References

Alvarez, L. H. R. (2004). Stochastic forest stand value and optimal timber harvesting. *SIAM Journal on Control and Optimization*, 42(6):1972–1993.

Aubin, J. P. (1982). *Mathematical Methods of Game and Economic Theory*. North-Holland, Amsterdam, The Netherlands.

Azimzadeh, P. (2019). A zero-sum stochastic differential game with impulses, precommitment, and unrestricted cost functions. *Applied Mathematics & Optimization*, 79(2):483–514.

Aïd, R., Basei, M., Callegaro, G., Campi, L., and Vargiolu, T. (2020). Nonzero-sum stochastic differential games with impulse controls: A verification theorem with applications. *Mathematics of Operations Research*, 45(1):205–232.

Baccarin, S. (2009). Optimal impulse control for a multidimensional cash management system with generalized cost functions. *European Journal of Operational Research*, 196(1):198 – 206.

Başar, T. and Olsder, G. J. (1999). *Dynamic Noncooperative Game Theory, 2nd edition*. Society for Industrial and Applied Mathematics, Philadelphia, PA.

Başar, T., Haurie, A., and Zaccour, G. (2018). Nonzero-Sum Differential Games. In Başar, T. and Zaccour, G., editors, *Handbook of Dynamic Game Theory*, pages 61–110. Springer International Publishing, Cham.

Basei, M. (2019). Optimal price management in retail energy markets: an impulse control problem with asymptotic estimates. *Mathematical Methods of Operations Research*, 89(3):355–383.

Basei, M., Cao, H., and Guo, X. (2019). Nonzero-sum stochastic games and mean-field games with impulse controls.

Bensoussan, A. and Lions, J.-L. (1982). *Contrôle Impulsionnel et Inéquations Quasi-Variationelles*. Bordas, Paris, France.

Bensoussan, A. and Lions, J.-L. (1984). *Impulse Control and Quasi-Variational Inequalities*. Gauthier-Villars, Paris, France.

Bensoussan, A. and Tapiero, C. S. (1982). Impulsive control in management: Prospects and applications. *Journal of Optimization Theory and Applications*, 37(4):419–442.

Berovic, D. P. and Vinter, R. B. (2004). The application of dynamic programming to optimal inventory control. *IEEE Transactions on Automatic Control*, 49(5):676–685.

Bertola, G., Runggaldier, W. J., and Yasuda, K. (2016). On classical and restricted impulse stochastic control for the exchange rate. *Applied Mathematics & Optimization*, 74(2):423–454.

Blaquière, A. (1977a). Differential games with piece-wise continuous trajectories. In Hagedorn, P., Knobloch, H. W., and Olsder, G. J., editors, *Differential Games and Applications*, pages 34–69. Springer, Berlin.

Blaquière, A. (1977b). Necessary and sufficient conditions for optimal strategies in impulsive control and application. In Aoki, M. and Morzzolla, A., editors, *New Trends in Dynamic System Theory and Economics*, pages 183–213. Academic Press, New York.

Cadenillas, A. and Zapatero, F. (1999). Optimal central bank intervention in the foreign exchange market. *Journal of Economic Theory*, 87(1):218 – 242.

Chahim, M., Grass, D., Hartl, R. F., and Kort, P. M. (2017). Product innovation with lumpy investment. *Central European Journal of Operations Research*, 25(1):159–182.

Chahim, M., Hartl, R. F., and Kort, P. M. (2012). A tutorial on the deterministic Impulse Control Maximum Principle: Necessary and sufficient optimality conditions. *European Journal of Operational Research*, 219(1):18–26.

Chikrii, A. A. and Matichin, I. (2005). Linear differential games with impulse control of players. *Proceedings of the Steklov Institute of Mathematics*, Suppl. 1:S68–S81.

Chikrii, A. A., Matychyn, I. I., and Chikrii, K. A. (2007). Differential games with impulse control. In Jørgensen, S., Quincampoix, M., and Vincent, T. L., editors, *Advances in Dynamic Game Theory. Annals of the International Society of Dynamic Games*, volume 9, pages 37–55. Birkhäuser, Boston.

Cosso, A. (2013). Stochastic Differential Games Involving Impulse Controls and Double-Obstacle Quasi-variational Inequalities. *SIAM Journal on Control and Optimization*, 51(3):2102–2131.

Crettez, B. and Hayek, N. (2014). Terrorists' eradication versus perpetual terror war. *Journal of Optimization Theory and Applications*, 160:679–702.

Dockner, E. J., Jorgensen, S., Long, N. V., and Sorger, G. (2000). *Differential Games in Economics and Management Science*. Cambridge University Press.

El Asri, B. (2013). Deterministic minimax impulse control in finite horizon: the viscosity solution approach. *ESAIM: Control, Optimisation and Calculus of Variations*, 19(1):63–77.

El Farouq, N., Barles, G., and Bernhard, P. (2010). Deterministic minimax impulse control. *Applied Mathematics and Optimization*, 61(3):353 – 378.

Engwerda, J. C. (2005). *LQ Dynamic Optimization and Differential Games*. Wiley, New York.

Etesami, S. R. and Başar, T. (2019). Dynamic games in cyber-physical security: An overview. *Dynamic Games and Applications*, 9(4):884–913.

Ferrari, G. and Koch, T. (2019). On a strategic model of pollution control. *Annals of Operations Research*, 275(2):297–319.

Geering, H. P. (1976). Continuous-Time Optimal Control Theory for Cost Functionals Including Discrete State Penalty Terms. *IEEE Transactions on Automatic Control*, 21(6):866–869.

Grames, J., Grass, D., Kort, P. M., and Prskawetz, A. (2019). Optimal investment and location decisions of a firm in a flood risk area using impulse control theory. *Central European Journal of Operations Research*, 27(4):1051–1077.

Haurie, A., Krawczyk, J. B., and Zaccour, G. (2012). *Games and Dynamic Games*. World Scientific, Singapore.

Jørgensen, S. and Zaccour, G. (2003). A differential game of retailer promotions. *Automatica*, 39(7):1145 – 1155.

Leander, R., Lenhart, S., and Protopopescu, V. (2015). Optimal control of continuous systems with impulse controls. *Optimal Control Applications and Methods*, 36(4):535–549.

Léonard, D. and Long, N. v. (1992). *Optimal Control Theory and Static Optimization in Economics*. Cambridge University Press.

Novak, A., Feichtinger, G., and Leitmann, G. (2010). A Differential Game Related to Terrorism: Nash and Stackelberg Strategies. *Journal of Optimization Theory and Applications*, 144:533–555.

Perera, S., Gupta, V., and Buckley, W. (2020). Management of online server congestion using optimal demand throttling. *European Journal of Operational Research*, 285(1):324 – 342.

Reddy, P. V., Wrzaczek, S., and Zaccour, G. (2016). Quality effects in different advertising models - An impulse control approach. *European Journal of Operational Research*, 255(3):984–995.

Sadana, U., Reddy, P. V., and Zaccour, G. (2021). Nash equilibria in nonzero-sum differential games with impulse control. Accepted by European Journal of Operational Research.

Sobiesiak, L. A. and Damaren, C. J. (2014). Linear quadratic optimal control for systems with continuous and impulsive inputs. In *53rd IEEE Conference on Decision and Control*, pages 5071–5076.

Taynitskiy, V., Gubar, E., and Zhu, Q. (2019). Optimal Impulse Control of SIR Epidemics Over Scale-Free Networks. In Song, J. B., Li, H., and Coupechoux, M., editors, *Game Theory for Networking Applications*, pages 119–129. Springer International Publishing, Cham.

Chapter 3

Sampled-data Nash equilibria in differential games with impulse control

Abstract

We study a class of deterministic two-player nonzero-sum differential games where one player uses piecewise-continuous controls to affect the continuously evolving state while the other player uses impulse controls at certain discrete instants of time to shift the state from one level to another. The state measurements are made at some given instants of time, and players determine their strategies using the last measured state value. We provide necessary conditions for the existence of sampled-data Nash equilibrium for a general class of differential games with impulse controls. We specialize our results for a scalar linear-quadratic differential game, and show that the equilibrium impulse timing can be obtained by determining a fixed point of a Riccati like system of differential equations with jumps coupled with a system of non-linear equality constraints. By reformulating our problem as a constrained non-linear optimization problem, we compute the equilibrium timing and level of impulses. We find that the equilibrium piecewise-continuous control is a linear function of the last measured state value. For linear-state differential games, we obtain analytical characterizations of equilibrium number, timing and levels of impulses in terms of the problem data, and provide an extension of our results for the case with piecewise-constant time-

varying problem parameters. In particular, there can be at most one impulse in the game when the problem parameters are fixed while each sampling interval can contain at most one impulse when the problem parameters differ between the sampling intervals. Using a numerical example, we illustrate our results.

3.1 Introduction

Recently, there has been renewed interest in the study of differential games with impulse controls where the state is controlled by two players, at least one of whom can affect the continuously evolving state variable at certain discrete instants of time only (Aïd et al., 2020; Ferrari and Koch, 2019; Sadana et al., 2020). The number and timing of interventions besides their level are also decision variables in the game. This allows for studying dynamic interactions in option pricing (El Farouq et al., 2010), pollution regulation (Ferrari and Koch, 2019), exchange rate interventions (Aïd et al., 2020), cybersecurity (Sadana et al., 2021), and related problems. A solution concept for these games involves determining the Nash equilibrium which depends on the information that is available to the players when they make their decisions (Başar and Olsder, 1999). Nash equilibrium in differential games with impulse controls have been obtained under two information structures, namely, open-loop and feedback information structures. In the open-loop information structure, the equilibrium controls of the players are obtained assuming that players have access to only the initial state, whereas with the feedback information structure, players make their decisions using the state measurements at each instant of time in the game. One limitation of using open-loop strategies is that they are not strongly time-consistent (Başar and Olsder, 1999; Başar, 1989), whereas the feedback equilibrium strategies require state measurements to be made at each instant of time in the game. In many real-world problems, for instance, economic data from the surveys, position of players in pursuit-evasion games, quality of goods, the state measurement is costly. As a result, state information is available to the players at the (discrete) sampling instants only, and the players determine their sampled-data controls (Simaan and Cruz Jr., 1973; Başar, 1991), using the previous state measurements. To the best of our knowledge, Nash equilibrium in differential games with impulse controls and sampling has not been studied in the literature.

In Simaan and Cruz Jr. (1973), the authors introduced a deterministic two-player nonzero-sum differential game where state measurement is made at discrete instants of time, and both players use piecewise-continuous strategies. The sampled-data controls of the players are assumed to be functions of the last measured state value, and players implement open-loop controls between the sampling instants. The authors showed that the equilibrium of linear-quadratic differential games can be obtained by solving a system of Riccati equations coupled with a system of differential equations that determine the terminal conditions on the Riccati equations. In Başar (1980), the author studies a stochastic linear-quadratic differential game where players have access to the sampled-data state information as well as the sampling times. A zero-sum linear-quadratic differential game with linear time-varying parameters was studied in Başar (1991) where it is shown that the optimal minimax sampled-data controller can be obtained by solving a generalized Riccati-differential equation. In Başar (1995), the author provided a characterization of the minimax controller of a switching system with sampled state information. In contrast to the aforementioned research that deals with piecewise-continuous controls, Drăgan et al. (2019) derived the Nash equilibrium of the stochastic linear-quadratic differential game assuming that the admissible strategies are constant between the state measurements.

In this paper, we consider a general class of deterministic two-player nonzero-sum differential games where the two players are endowed with different kinds of controls (discrete and piecewise-continuous). In particular, Player 1 uses piecewise-continuous controls to affect the continuous evolution of state whereas Player 2 uses impulse controls to shift the state value instantaneously from one level to another at the impulse instants that are endogenously determined by Player 2 in addition to the number of impulse instants. The more general case with both players using continuous and impulse controls can be easily studied using our model. However, for the application of our work in problems involving regulation and cybersecurity, we can restrict our focus to our canonical game model with one player using piecewise-continuous controls and the other player using impulse controls.

The objectives of this research are three-fold: First, we aim to provide necessary conditions for the existence of Nash equilibrium. Our second objective is to specialize our results for scalar linear-quadratic differential games (LQDGs) which are widely used in economics, engineering and management domains (see Başar and Olsder, 1999; Haurie et al., 2012; Başar et al., 2018) as they allow the possibility to model real-world problems involving non-linear returns to scale. Also, linear dynamics can approximate sufficiently well the non-linear dynamics, at least in some applications. Third, we aim to determine analytical solutions for equilibrium number, timing and levels of impulses in scalar linear-state differential games (see Başar and Olsder, 1999; Dockner et al., 2000; Engwerda, 2005; Haurie et al., 2012) where we restrict the payoff functions to be linear in state, and state dynamics to be linear in both state and controls of the players.

Our contributions can be summarized as follows:

(i) For the first time, our paper provides necessary conditions for the existence of Nash equilibrium in a differential game with impulse controls where the players' strategies are functions of the state values measured at certain discrete time instants; see Theorem 3.1.

(ii) For the case of LQDGs with exogenously given impulse instants, Theorem 3.2 provides a system of Riccati like equations with jumps which characterize the sampled-data Nash equilibrium.

(iii) For LQDGs with a given number of impulses in each sampling interval, Theorem 3.3 shows that the equilibrium timing of impulses can be obtained as a solution of a system of Riccati equations (with jumps) provided that the impulse instants satisfy a system of non-linear equality constraints. In particular, we show that an impulse occurs when the state trajectory hits a time-varying function of the gradient of the value function of Player 2.

(iv) In Theorem 3.4, we show that there can be at most one impulse in the sampled-data Nash equilibrium of a scalar linear-state differential game. When the problem parameters are piecewise-constant functions of time, we show that in the scalar linear-state differential game, the number of impulses is at most equal to the number of sampling intervals; see Theorem 3.5.

The rest of this paper is organized as follows: In Section 3.2, we introduce our canonical two-player differential game model. Section 3.3 provides necessary conditions for the existence of sampled-data Nash equilibrium for our canonical model. In Section 3.4, we specialize the necessary conditions to a scalar linear-quadratic differential game. We further specialize our results to a scalar linear-state differential game in Section 3.5, and also provide an extension of our game to problems with time-varying parameters. Further, we illustrate the theoretical results using a numerical example in Section 3.6. Finally, Section 3.7 provides concluding remarks, and the paper ends with an appendix, which details the proof of Theorem 3.2.

3.2 Model

In this paper, we consider a deterministic two-player differential game of finite duration $T < \infty$ where both players can affect a continuously evolving state variable $x(t) \in \mathbb{R}^n$ to maximize their payoffs. However, the two players are endowed with different kinds of controls. Player 1 can continuously influence the dynamics of the state variable using her piecewise-continuous controls $u(t) \in \Omega_u$ while Player 2 can intervene and cause jumps in the state variable at certain discrete instants of time τ_i $(i = 1, 2, \cdots, k)$. We assume that Ω_u is a bounded and convex open subset of $\mathbb{R}^{m_1}$. When Player 2 does not intervene in the game, the state variable is continuous and its dynamics are controlled entirely by Player 1 so that the state variable evolves as follows:

$$\dot{x}(t) = f(x(t), u(t)),\ x(0^-) = x_0,\ \text{for}\ t \neq \{\tau_1, \tau_2, ..., \tau_k\}, \tag{3.1}$$

where $f : \mathbb{R}^n \times \Omega_u \to \mathbb{R}^n$, the initial value of state variable is given by $x_0 \in \mathbb{R}^n$ (a known parameter), $x(\tau_i^-) = \lim_{t\uparrow\tau_i} x(t)$, $x(\tau_i^+) = \lim_{t\downarrow\tau_i} x(t)$, and 0^- denotes the time instant just before 0. At the impulse instants τ_i, Player 2 intervenes in the game to shift the state from $x(\tau_i^-)$ to $x(\tau_i^+)$ by using an impulse of size $v_i \in \Omega_v$, that is,

$$x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i^-), v_i),\ i \in \{1, 2, \cdots, k\}, \tag{3.2}$$

where $g : \mathbb{R}^n \times \Omega_v \to \mathbb{R}^n$. We assume that Ω_v is a bounded and convex open subset of $\mathbb{R}^n$. The number of impulses $k \in \mathbb{N}$ (the set of natural numbers), and timing of impulses τ_i are decision variables of Player 2 in addition to the levels of impulses. The impulse controls are denoted by $\tilde{v} = \{(\tau_i, v_i),\ i = \{1, 2, \cdots, k\}, k\}$.

In this differential game, Player 1 maximizes the following objective:

$$J_1(x_0, u(\cdot), \tilde{v}) = \int_0^T F_1(x(t), u(t))dt + \sum_{i=1}^{k} G_1(x(\tau_i^-), v_i) + S_1(x(T^+)), \tag{3.3}$$

and Player 2 uses the impulse controls (τ_i, v_i) to maximize the objective

$$J_2(x_0, u(\cdot), \tilde{v}) = \int_0^T F_2(x(t), u(t))dt + \sum_{i=1}^{k} G_2(x(\tau_i^-), v_i) + S_2(x(T^+)), \tag{3.4}$$

where $F_1,\ F_2 : \mathbb{R}^n \times \Omega_u \to \mathbb{R}$, $G_1,\ G_2 : \mathbb{R}^n \times \Omega_v \to \mathbb{R}$, and $S_1,\ S_2 : \mathbb{R}^n \to \mathbb{R}$. For Player i, F_i denotes the running payoff, G_i denotes the intervention cost at the impulse instants, and S_i is the terminal payoff.

In a differential game, the Nash equilibrium depends on the state information that the players use to determine their strategies (see Başar and Olsder, 1999; Haurie et al., 2012). We assume that the state measurement is made at certain discrete instants of time $t_n, n \in \{1, 2, \cdots, N\}$, with the corresponding state values denoted by $x_1, x_2, \cdots, x_N$ such that $0 = t_1 < t_2 < \cdots < t_{N-1} < t_N = T$. The sampled-data controls of Player 1 are given by

$$u(t) = \gamma(t; x(t_n)) \in \Omega_u,\ \text{for}\, t_n \leq t < t_{n+1},\ n \in \mathcal{N}' = \{1, 2, \cdots, N-1\},\ \gamma \in \Gamma, \tag{3.5}$$

where $\gamma : [t_n, t_{n+1}] \times \mathbb{R}^n \to \Omega_u$ is a sampled-data state feedback controller of Player 1 and the strategy set of Player 1 is denoted by Γ. Similarly, the impulse levels of Player 2 are given by

$$v_{i,n} = \delta(\tau_{i,n}; x(t_n)) \in \Omega_v,\ \text{for}\, t_n \leq \tau_{i,n} < t_{n+1},\ n \in \mathcal{N}',\ \delta \in \Delta, \tag{3.6}$$

where $\delta : [t_n, t_{n+1}] \times \mathbb{R}^n \to \Omega_u$ is a sampled-data state feedback controller for Player 2 and Δ denotes the strategy set of Player 2.

The objective functions of the players over the sub-interval $[t_n, T]$, initialized at the sampling instant t_n with the corresponding state $x(t_n) = x_n$ are given by

$$\begin{aligned} J_1(x_n, \gamma_{[t_n,T]}, \delta_{[t_n,T]}) = \sum_{j=n}^{N-1} \Big(\int_{t_j}^{t_{j+1}} & F_1(x(t), \gamma(t; x(t_j)))dt \\ & + \sum_{i=1}^{k_j} \mathbb{1}_{\tau_{i,j} \geq t_n} G_1(x(\tau_{i,j}^-), \delta(\tau_{i,j}; x(t_j))) \Big) + S_1(x(T^+)), \end{aligned} \tag{3.7a}$$

$$J_2(x_n, \gamma_{[t_n,T]}, \delta_{[t_n,T]}) = \sum_{j=n}^{N-1} \Big(\int_{t_j}^{t_{j+1}} F_2(x(t), \gamma(t; x(t_j)))dt + \sum_{i=1}^{k_j} \mathbb{1}_{\tau_{i,j} \geq t_n} G_2(x(\tau_{i,j}^-), \delta(\tau_{i,j}; x(t_j))) \Big) + S_2(x(T^+)), \tag{3.7b}$$

where the strategies $\gamma_{[t_n,T]}$ and $\delta_{[t_n,T]}$ are restrictions of γ and δ to the interval $[t_n, T]$, and $\Gamma_{[t_n,T]}$ and $\Delta_{[t_n,T]}$ denote the corresponding admissible strategy sets of Player 1 and Player 2, respectively. The state dynamics are given by

$$\dot{x}(t) = f(x(t), \gamma(t; x_n)),\ x(t_n^-) = x_n,\ \text{for}\ t_n \leq t < t_{n+1},\ n \in \mathcal{N}', \tag{3.7c}$$

$$x(\tau_{i,n}^+) - x(\tau_{i,n}^-) = g(x(\tau_{i,n}^-), \delta(\tau_{i,n}; x_n)),\ \text{for}\ i \in \mathcal{I}^n = \{1, 2, \cdots, k_n\},\ n \in \mathcal{N}', \tag{3.7d}$$

where k_n denotes the equilibrium number of impulses in the sampling interval $[t_n, t_{n+1}]$. From (3.7a)-(3.7b), it is clear that each player can influence the payoff of their opponent directly through their controls, and indirectly by changing the state variable.

Remark 3.1 *The above canonical differential game model (3.7a-3.7d) can be used to study problems in cybersecurity and pollution regulation where the running payoff of one player, say Player 1, decreases with state and Player 2's running payoff is increasing with state. Player 1 continuously invests in reducing the state except at the impulse instants wherein Player 2 intervenes in the game to instantaneously shift the state to a higher value. Consequently, Player 1 incurs a state-dependent cost at the impulse instant.*

Clearly, the admissible controls in the aforementioned real-world applications satisfy the following definition:

Definition 3.1 $(\tau_{i,n}, v_{i,n})$, $i \in \mathcal{I}^n$, $n \in \mathcal{N}'$, *is an admissible impulse control of Player 2 if the impulse instants satisfy the following increasing monotone sequence property:*

$$t_n < \tau_{1,n} < \tau_{2,n} < \cdots < \tau_{k_n,n} < t_{n+1}, \tag{3.8}$$

where $k_n < \infty$, $v_{i,n} \neq 0$, *and it is assumed that the impulse instants are interior, that is,* $\tau_{i,n} \in (t_n, t_{n+1})$.

In this paper, we seek to determine the sampled-data Nash equilibrium of the differential game (3.7a-3.7d), which is defined as follows:

Definition 3.2 *The strategy profile* (γ^*, δ^*) *is a sampled-data Nash equilibrium of the differential game (3.7a-3.7d), if the restrictions of* γ^* *and* δ^**, denoted by* $\gamma^*_{[t_n,T]}$ *and* $\delta^*_{[t_n,T]}$*, to any subgame that starts at the sampling time* t_n *with state measurement* x_n *satisfy the following inequalities:*

$$J_1(x_n, \gamma^*_{[t_n,T]}, \delta^*_{[t_n,T]}) \geq J_1(x_n, \gamma_{[t_n,T]}, \delta^*_{[t_n,T]}),\ \forall \gamma_{[t_n,T]} \in \Gamma_{[t_n,T]}, \tag{3.9a}$$

$$J_2(x_n, \gamma^*_{[t_n,T]}, \delta^*_{[t_n,T]}) \geq J_2(x_n, \gamma^*_{[t_n,T]}, \delta_{[t_n,T]}),\ \forall \delta_{[t_n,T]} \in \Delta_{[t_n,T]}. \tag{3.9b}$$

Remark 3.2 *The sampled-data Nash equilibrium strategies of the differential game (3.7a-3.7d) for* $t \in [0, T]$ *when restricted to* $[t_n, T]$ *are also the Nash equilibrium strategies of the subgame that starts at* t_n*. As a result, the sampled-data Nash equilibrium strategies are strongly time-consistent (Başar, 1989) if the perturbation of state can occur only at the sampling instants* t_n*,* $n \in \mathcal{N} = \{1, 2, \cdots, N\}$*. At all other time instants, that is,* $t \neq t_n, n \in N$*, the sampled-data Nash equilibrium strategies are weakly time-consistent (Başar, 1989).*

Remark 3.3 *When sampling is done at the initial and final time only, then the sampled-data Nash equilibrium coincides with the open-loop Nash equilibrium of a differential game. It is shown in Simaan and Cruz Jr. (1973) that the sampled-data equilibrium controls approach the closed-loop controls as the number of sampling intervals increases.*

3.3 Necessary conditions

In this section, we derive the necessary conditions for the existence of sampled-data Nash equilibrium in differential games with impulse controls.

The approach to determine the sampled-data Nash equilibrium can be summarized as follows. Suppose the sampling instants are given by $t_1, t_2, \cdots, t_N$. For $t \in [t_n, t_{n+1}]$, players use open-loop strategies $\gamma^*(t; x_n)$ and $\delta^*(t; x_n)$, which are functions of last measured state value x_n, that is, for any given initial state x_n, Player 1 determines the open-loop controls in the sampling interval and Player 2 determines the equilibrium number, timing and levels of impulses. The payoff of each player at $(t_n, x(t_n))$ is a salvage value for the open-loop game between t_{n-1} and t_n. Therefore, starting from the last sampling interval $[t_{N-1}, T]$ with salvage values S_1 and S_2, we can recursively obtain the equilibrium strategies for all the sampling intervals $[t_n, t_{n+1}]$, $n \in \mathcal{N}'$.

First, we define the Hamiltonians of the two players that will be used in the necessary conditions for the existence of sampled-data Nash equilibrium. The continuous Hamiltonians of Player 1 and Player 2 are given by

$$H_1(x(t), u(t), \lambda_1(t)) = F_1(x(t), u(t), \lambda_1(t)) + \lambda_1(t)^T f(x(t), u(t)), \tag{3.10}$$

$$H_2(x(t), u(t), \lambda_2(t)) = F_2(x(t), u(t), \lambda_2(t)) + \lambda_2(t)^T f(x(t), u(t)), \tag{3.11}$$

where $\lambda_1(.)$ and $\lambda_2(.)$ denote the co-states of Player 1 and Player 2, respectively. The impulse Hamiltonian of Player 2 is given by

$$H_2^I(x(t), v, \lambda_2(t)) = G_2(x(t), v) + \lambda_2(t)^T g(x(t), v). \tag{3.12}$$

Given the strategies, γ and δ, the value-to-go functions of Player 1 and Player 2 at the sampling instants $t_{n+1},\ n \in \mathcal{N}'$ are given by

$$\begin{aligned} V_1(t_{n+1}, x_{n+1}) = \sum_{j=n+1}^{N-1} \Big(&\int_{t_j}^{t_{j+1}} F_1(x(t), \gamma(t; x_j))dt \\ &+ \sum_{i=1}^{k_j} \mathbb{1}_{\tau_{i,j} \geq t_{n+1}} G_1(x(\tau_{i,j}^-), \delta(\tau_{i,j}; x_j)) \Big) + S_1(x(T)), \end{aligned} \tag{3.13a}$$

$$\begin{aligned} V_2(t_{n+1}, x_{n+1}) = \sum_{j=n+1}^{N-1} \Big(&\int_{t_j}^{t_{j+1}} F_2(x(t), \gamma(t; x_j))dt \\ &+ \sum_{i=1}^{k_j} \mathbb{1}_{\tau_{i,j} \geq t_{n+1}} G_2(x(\tau_{i,j}^-), \delta(\tau_{i,j}; x_j)) \Big) + S_2(x(T)), \end{aligned} \tag{3.13b}$$

with $V_1(T, x(T)) = S_1(x(T))$, and $V_2(T, x(T)) = S_2(x(T))$. We denote the equilibrium payoffs of Player 1 and Player 2 at t_{n+1} by $V_1^*(t_{n+1}, x_{n+1})$ and $V_2^*(t_{n+1}, x_{n+1})$, respectively.

To derive a set of necessary conditions for the existence of Nash equilibrium, we make the following assumptions:

Assumption 3.1 *(a) The function $f : \mathbb{R}^n \times \Omega_u \to \mathbb{R}^n$ is Lipschitz continuous in x for all u.*

(b) Between the sampling instants, the functions F_1, F_2, G_1, G_2 are continuous, and have continuous partial derivatives with respect to their arguments. The value-to-go functions V_1 and V_2 are continuous, and have continuous partial derivatives with respect to the state at the sampling instants.

The following theorem gives the necessary conditions for the existence of sampled-data Nash equilibrium of the differential game (3.7a-3.7d).

Theorem 3.1 *Suppose the sampling instants are given by $t_1, t_2, \cdots, t_N$ with $0 = t_1 < t_2 < \cdots < t_N = T$, and Assumption 3.1 holds. Let (γ^*, δ^*) be the sampled-data Nash equilibrium of the differential game described by (3.7a-3.7d). Then, there exist piecewise continuous and piecewise differentiable functions $\lambda_1(.)$ and $\lambda_2(.)$ with $\lambda_1(t) \in \mathbb{R}^n$ and $\lambda_2(t) \in \mathbb{R}^n$ such that the following conditions hold for $t \in [t_n, t_{n+1}), n \in \mathcal{N}'$:*
The equilibrium control of Player 1 satisfies

$$u^*(t) = \arg\max_{u \in \Omega_u} H_1(x^*(t), u(t), \lambda_1(t)), \forall t \notin \mathcal{T}^n = \{\tau^*_{1,n}, \tau^*_{1,n}, \cdots, \tau^*_{k^*_n,n}\}. \tag{3.14a}$$

At the impulse instant $\tau^*_{i,n}$, $i \in \mathcal{I}^n$, the equilibrium control of Player 2 satisfies

$$v^*_{i,n} = \arg\max_{v_{i,n} \in \Omega_v} H^I_2(x^*(\tau^{*-}_{i,n}), v_{i,n}, \lambda_2(\tau^{*+}_{i,n})). \tag{3.14b}$$

The equilibrium strategies of Player 1 and Player 2 are given by $\gamma^*(t; x_n) = u^*(t), \forall t \in [t_n, t_{n+1}]$, $t \notin \mathcal{T}^n$ and $\delta^*(\tau^*_{i,n}; x_n) = v^*_{i,n}$, $\forall i \in \mathcal{I}^n$.
The maximized Hamiltonian and impulse Hamiltonian functions are given by

$$H^*_1(x^*(t), \lambda_1(t)) = H_1(x^*(t), u^*(t), \lambda_1(t)),\ \forall t \notin \mathcal{T}^n, \tag{3.14c}$$

$$H^{I^*}_2(x^*(\tau^{*-}_{i,n}), \lambda_2(\tau^{*+}_{i,n})) = H^I_2(x^*(\tau^{*-}_{i,n}), v^*_{i,n}, \lambda_2(\tau^{*+}_{i,n})),\ i \in \mathcal{I}^n, \tag{3.14d}$$

the equilibrium state and co-state equations satisfy for $t \notin \mathcal{T}^n$,

$$\dot{x}^*(t) = f(x^*(t), u^*(t)),\ x^*(t_n) = x_n, \tag{3.14e}$$

$$\dot{\lambda}_1(t) = -H^*_{1x}(x^*(t), \lambda_1(t)),\ \lambda_1(t_{n+1}) = \frac{\partial V^*_1(t_{n+1}, x(t_{n+1}))}{\partial x}, \tag{3.14f}$$

$$V^*_1(T, x(T)) = S_1(x(T)),$$

$$\dot{\lambda}_2(t) = -H^*_{2x}(x^*(t), u^*(t), \lambda_2(t)),\ \lambda_2(t_{n+1}) = \frac{\partial V^*_2(t_{n+1}, x(t_{n+1}))}{\partial x}, \tag{3.14g}$$

$$V^*_2(T, x(T)) = S_2(x(T)),$$

the jumps in the state and co-state variables satisfy for $i \in \mathcal{I}^n$

$$x^*(\tau^{*+}_{i,n}) = x^*(\tau^{*-}_{i,n}) + g(x^*(\tau^{*-}_{i,n}), v^*_{i,n}), \tag{3.14h}$$

$$\lambda_1(\tau^{*-}_{i,n}) = (I + (g_x(x^*(\tau^{*-}_{i,n}), v^*_{i,n}))^T)\lambda_1(\tau^{*+}_{i,n}) + G_{1x}(x^*(\tau^{*-}_{i,n}), v^*_{i,n}), \tag{3.14i}$$

$$\lambda_2(\tau^{*-}_{i,n}) = \lambda_2(\tau^{*+}_{i,n}) + H^{I*}_{2x}(x^*(\tau^{*-}_{i,n}), \lambda_2(\tau^{*+}_{i,n})), \tag{3.14j}$$

and the following Hamiltonian continuity condition holds:

$$H_2(x^*(\tau_{i,n}^{*+}), u^*(\tau_{i,,n}^{*+}), \lambda_2(\tau_{i,n}^{*+})) = H_2(x^*(\tau_{i,n}^{*-}), u^*(\tau_{i,n}^{*-}), \lambda_2(\tau_{i,n}^{*-})). \tag{3.14k}$$

Proof. For $t \in [t_n, t_{n+1}]$, Player 1 and Player 2 play their open-loop Nash equilibrium strategies, $\gamma^*(t; x_n)$ and $\delta^*(\tau_{i,n}; x_n)$, that depend on the last measured state value x_n. The salvage values of the two players at t_{n+1} are given by (3.13a) and (3.13b).

Given the equilibrium strategy $\delta^*(\tau_{i,n}^*, x_n)$ of Player 2 in the sampling interval $[t_n, t_{n+1}]$, Player 1 solves a non-standard optimal control problem given in (3.9a) due to jumps in the state and the additional cost at the impulse instant. Suppose Assumption 3.1 holds. Then, the optimality conditions for Player 1 are given in (3.14a), (3.14e), (3.14f), (3.14h),
(3.14i) (see Geering, 1976; Sadana et al., 2021), with co-state at t_{n+1} given by the gradient of the equilibrium payoff of Player 1 at t_{n+1}. Next, for Player 1's open-loop equilibrium strategy, $\gamma^*(t; x_n)$ in $[t_n, t_{n+1}]$, Player 2 solves the impulse optimal control problem (3.9b). The necessary conditions for the existence of the impulse controls follow from Blaquière (1977a,b), Chahim et al. (2012), and are given by (3.14b), (3.14e), (3.14h), (3.14g), (3.14j), (3.14k), where the co-state at t_{n+1} is given by the gradient of the equilibrium payoff of Player 2 at t_{n+1}. ■

The necessary conditions yield the candidates for the sampled-data Nash equilibrium. In each sampling interval, the players use open-loop Nash equilibrium strategies, and the game is solved using backward translation starting from the last sampling interval. Consequently, if the sufficient conditions for the open-loop Nash equilibrium are satisfied in each sampling interval, then the candidate solutions identified by using the necessary conditions are indeed the sampled-data Nash equilibrium strategies.

Sufficient conditions for the existence of sampled-data Nash equilibrium for the differential game described by (3.7a-3.7d) are given as follows:

Proposition 3.1 *(Sadana et al., 2021, Theorem 3) Let Assumption 3.1 hold. Suppose in each sampling interval $[t_n, t_{n+1}], n \in \mathcal{N}'$, the initial state is given by x_n, and there exist feasible solutions $(\gamma^*(t; x_n), \delta^*(\tau_{i,n}^*; x_n))$ with corresponding state trajectory $x^*(.)$, and co-state trajectories $\lambda_1(.)$ and $\lambda_2(.)$, such that the conditions given in Theorem 3.1 are* *satisfied. Also, if in each sampling interval, the maximized Hamiltonian* $H_1^*(x(t), \lambda_1(t))$ *of Player 1 is concave in* $x(t)$ *for all* $\lambda_1(t)$*, the Hamiltonian* $H_2(x(t), u^*(t), \lambda_2(t))$ *of Player 2 is concave in* $x(t)$*, the value-to-go functions for Player 1 and Player 2 given by* (3.13a) *and* (3.13b) *are concave in* $x(t_{n+1})$*,* $G_1(x(t), v) + \lambda_1^T g(x(t), v)$ *is concave in* $x(t)$*, and the impulse Hamiltonian* $H_2^I(x(t), v, \lambda_2(t))$ *of Player 2 is concave in* $(x(t), v)$*, then* (γ^*, δ^*)*, obtained by concatenating the (open-loop) strategies* $(\gamma^*(t; x_n), \delta^*(\tau_{i,n}^*; x_n))$ *for* $t \in [t_n, t_{n+1}]$*, are indeed the sampled-data Nash equilibrium strategies of the differential game described by (3.7a-3.7d).*

3.4 Scalar linear-quadratic differential game

In this section, we specialize our results in Theorem 3.1 to a one-dimensional linear-quadratic differential game with impulse controls, where state measurements are made at the sampling instants t_n, $n \in \mathcal{N} = \{1, 2, \cdots, N\}$ such that $0 = t_1 < t_2 < \cdots < t_N = T$.

We study the following scalar linear-quadratic differential game with impulse controls (referred to as iLQDG from here on):

$$\begin{aligned}
\text{(iLQDG)}\ J_1(x_0, u(\cdot), \tilde{v}) =& \frac{1}{2}\Big[\sum_{n=1}^{N-1}\Big(\int_{t_n}^{t_{n+1}} \big(h_1 x(t)^2 + 2w_1 x(t) + c_u u(t)^2\big)\, dt \\
&+ \sum_{i=1}^{k_n}\big(z_1 x(\tau_{i,n}^-)^2 + 2d_1 x(\tau_{i,n}^-)\big)\Big) \\
&+ f_1 x(T)^2 + 2s_1 x(T)\Big], \qquad (3.15)\\
J_2(x_0, u(\cdot), \tilde{v}) =& \sum_{n=1}^{N-1}\Big(\int_{t_n}^{t_{n+1}} w_2 x(t) dt + \sum_{i=1}^{k_n}\left(\frac{1}{2} c_v v_{i,n}^2\right)\Big) + s_2 x(T),\\
\dot{x}(t) =& a x(t) + b u(t),\ \forall t \notin \mathcal{T}^n,\ n \in \mathcal{N}',\ x(0) = x_0,\\
x(\tau_{i,n}^+) =& x(\tau_{i,n}^-) + g v_{i,n},\ \forall i \in \mathcal{I}^n = \{1, 2, \cdots, k_n\},\ n \in \mathcal{N}',
\end{aligned}$$

where $b \neq 0$, $g \neq 0$, $c_u < 0$, $c_v < 0$, and the state at the sampling instants $t_1, t_2, \cdots, t_N$ is denoted by $x_1, x_2, \cdots, x_N$.
We make the following assumptions on the equilibrium controls of the players:

Assumption 3.2 *In each sampling interval, Player 1's strategy space* $\Gamma_{[t_n, t_{n+1}]}$ *is the set of locally square-integrable functions, that is,*

$$\Gamma_{[t_n,t_{n+1}]} := \left\{ u(t) \in \mathbb{R},\ t \in [t_n, t_{n+1}] \,\middle|\, \int_{t_n}^{t_{n+1}} u^T(t) u(t) dt < \infty \right\}, \qquad (3.16)$$

and Player 2's controls satisfy Definition 3.1.

Assumption 3.3 *The equilibrium controls $u^*(t)$ of Player 1 and equilibrium impulse levels v_i^* of Player 2 lie in the interior of the control sets Ω_u and Ω_v, respectively.*

3.4.1 Necessary conditions

Before considering the case where the number, timing and levels of impulses are determined by Player 2, we consider the differential game (3.15) with exogenously given impulse instants.

Theorem 3.2 *Let $t_1, t_2, \cdots, t_N$ denote the sampling instants, and suppose that Assumptions 3.2 and 3.3 hold. Let the equilibrium impulse instants be given by $\tau_{i,n}^*, \forall i \in \mathcal{I}^n = \{\tau_1^*, \tau_2^*, \cdots, \tau_{k_n^*,n}^*\}, n \in \mathcal{N}' = \{1, 2, \cdots, N-1\}$. Then γ^* and δ^* are the equilibrium strategies of Player 1 and Player 2, respectively if the following Riccati system for $n \in \mathcal{N}'$ has a solution with no finite escape time in all the sampling intervals $[t_n, t_n + 1]$:*

$$\dot{\alpha}_{1,n}(t) = -2\alpha_{1,n}(t)a + \frac{b^2}{c_u}\alpha_{1,n}(t)^2 - h_1,\ \forall t \notin \mathcal{T}^n, \tag{3.17a}$$

$$\alpha_1(t_{n+1}) = p_{1,n+1}(t_{n+1}),\ \alpha_{1,N}(T) = f_1,$$

$$\dot{\beta}_{1,n}(t) = \beta_{1,n}(t)\left(\frac{b^2}{c_u}\alpha_{1,n}(t) - a\right) - w_1,\ \forall t \notin \mathcal{T}^n, \tag{3.17b}$$

$$\beta_1(t_{n+1}) = q_{1,n+1}(t_{n+1}), \beta_{1,N}(T) = s_1,$$

$$\alpha_{1,n}(\tau_{i,n}^{*-}) = \alpha_{1,n}(\tau_{i,n}^{*+}) + z_1,\ \forall i \in \mathcal{I}^n, \tag{3.17c}$$

$$\beta_{1,n}(\tau_{i,n}^{*-}) = \beta_{1,n}(\tau_{i,n}^{*+}) - \alpha_{1,n}(\tau_{i,n}^{*+})\frac{g^2}{c_v}\lambda_2(\tau_{i,n}^{*+}) + d_1,\ \forall i \in \mathcal{I}^n, \tag{3.17d}$$

$$\begin{aligned}\dot{p}_{1,n}(t) = -h_1 - 2(a - \frac{b^2}{c_u}\alpha_{1,n}(t))p_{1,n}(t) \\ - \frac{b^2}{c_u}\alpha_{1,n}(t)^2,\ \forall t \notin \mathcal{T}^n,\ p_{1,N}(T) = f_1,\end{aligned} \tag{3.17e}$$

$$\begin{aligned}\dot{q}_{1,n}(t) = -w_1 + \frac{b^2}{c_u}p_{1,n}(t)\beta_{1,n}(t) - q_{1,n}(t)(a - \frac{b^2}{c_u}\alpha_{1,n}(t)) \\ - \frac{b^2}{c_u}\alpha_{1,n}(t)\beta_{1,n}(t),\ \forall t \notin \mathcal{T}^n,\ q_{1,N}(T) = s_1,\end{aligned} \tag{3.17f}$$

$$p_{1,n}(\tau_{i+1,n}^{*-}) = p_{1,n}(\tau_{i+1,n}^{*+}) + z_1,\ \forall i \in \mathcal{I}^n, \tag{3.17g}$$

$$q_{1,n}(\tau_{i+1,n}^{*-}) = -p_{1,n}(\tau_{i+1,n}^{*+})\frac{g^2}{c_v}\lambda_2(\tau_{i,n}^{*+}) + q_{1,n}(\tau_{i+1,n}^{*+}) + d_1,\ \forall i \in \mathcal{I}^n, \tag{3.17h}$$

$$p_{1,n}(t_{n+1}) = p_{1,n+1}(t_{n+1}),\ q_{1,n}(t_{n+1}) = q_{1,n+1}(t_{n+1}), \tag{3.17i}$$

$$\dot{q}_{2,n}(t) = -w_2 - (a - \frac{b^2}{c_u}\alpha_{1,n}(t))q_{2,n}(t), \forall t \notin \mathcal{T}^n,\ q_{2,N}(T) = s_2, \tag{3.17j}$$

$$\lambda_2(t_{n+1}) = q_{2,n}(t_{n+1}), \tag{3.17k}$$

$$\lambda_2(t) = -\frac{w_2}{a} + (\lambda_2(t_{n+1}) + \frac{w_2}{a})e^{a(t_{n+1}-t)},\ \forall t \in [t_n, t_{n+1}), \tag{3.17l}$$

$$q_{2,n}(\tau^{*-}_{i+1,n}) = q_{2,n}(\tau^{*+}_{i+1,n}),\ \forall i \in \mathcal{I}^n, \tag{3.17m}$$

$$q_{2,n}(t_{n+1}) = q_{2,n+1}(t_{n+1}),\ q_{2,N}(T) = s_2. \tag{3.17n}$$

The equilibrium strategies of Player 1 and Player 2 are given by

$$\begin{aligned}\gamma^*(t; x_n) = &-\frac{b}{c_u}\Big(\alpha_{1,n}(t)\big(\phi(t,\tau^{*+}_{i,n})(\phi(\tau^{*-}_{i,n}, t_n)x_n + gv^*_{i,n}\mathbb{1}_{t>\tau^*_{i,n}} \\ &+ \varphi(\tau^{*-}_{i,n}, t_n)) + \varphi(t, \tau^{*+}_{i,n})\big) + \beta_{1,n}(t)\Big), \forall t \in [\tau^*_{i,n}, \tau^*_{i+1,n}), i \in \mathcal{I}^n \cup \{0\},\end{aligned} \tag{3.18a}$$

$$\delta^*(\tau^*_{i,n}; x_n) = \frac{g}{c_v}\left(\frac{w_2}{a} - (\lambda_2(t_{n+1}) + \frac{w_2}{a})e^{a(t_{n+1}-\tau^*_{i,n})}\right), \tag{3.18b}$$

where $\tau^*_{0,n} := t_n,\ \tau^*_{k^*_n+1,n} := t_{n+1}$*, and* $\forall i \in \{0\} \cup \mathcal{I}^n$*,*

$$\dot{\phi}(t, \tau^*_{i,n}) = \left(a - \frac{b^2}{c_u}\alpha_{1,n}(t)\right)\phi(t,\tau^*_{i,n}), \forall t \in (\tau^*_{i,n}, \tau^*_{i+1,n}), \phi(\tau^*_{i,n}, \tau^*_{i,n}) = 1, \tag{3.19a}$$

$$\varphi(t, \tau^{*-}_{i,n}) = -\int_{\tau^{*-}_{i,n}}^{t} \phi(h, \tau^{*-}_{i,n})\frac{b^2}{c_u}\beta_{1,n}(h)dh, \forall t \in (\tau^*_{i,n}, \tau^*_{i+1,n}), \tag{3.19b}$$

$$\phi(\tau^{*-}_{i+1,n}, t_n) = \phi(\tau^{*-}_{i+1,n}, \tau^{*+}_{i,n})\phi(\tau^{*-}_{i,n}, t_n), \tag{3.19c}$$

$$\begin{aligned}\varphi(\tau^{*-}_{i+1,n}, t_n) = &\ \phi(\tau^{*-}_{i+1,n}, \tau^{*+}_{i,n})\varphi(\tau^{*-}_{i,n}, t_n) - \phi(\tau^{*-}_{i+1,n}, \tau^{*+}_{i,n})\frac{g^2}{c_v}\lambda_2(\tau^{*+}_{i,n}) \\ &+ \varphi(\tau^{*-}_{i+1,n}, \tau^{*+}_{i,n}).\end{aligned} \tag{3.19d}$$

Proof. See Appendix. ■

Remark 3.4 *Even when the timing of impulses is given, the Riccati like system of equations* (3.17a)–(3.17n) *differ from those obtained for classical differential games without impulse controls because of jumps in state and additional costs incurred by the players at the impulse instants.*

The above theorem characterizes the equilibrium with exogenously given impulse instants. If the number and timing of impulses are determined by Player 2, the impulse instants must satisfy the Hamiltonian continuity condition (3.14k) in addition to (3.17a)–(3.17n).

Theorem 3.3 *Suppose $t_1,\, t_2, \cdots, t_N$ are the sampling instants, and Assumptions 3.2 and 3.3 hold. Then $\tau^*_{i,n}$, $i \in \mathcal{I}^n$, $n \in \mathcal{N}'$ are the equilibrium impulse instants if*

$$\begin{aligned} x(\tau^*_{i,n}) =& \phi(\tau^{*-}_{i,n}, t_n)x_n + \varphi(\tau^{*-}_{i,n}, t_n) \\ =& \frac{\left(\frac{c_u g^2}{c_v b^2}(a q_{2,n+1}(t_{n+1}) + w_2)e^{a(t_{n+1}-\tau^*_{i,n})}\right) - d_1}{z_1}, \end{aligned} \tag{3.20a}$$

where $q_{2,n+1}(t_{n+1})$ is the gradient of the value function of Player 2 at the sampling instant t_{n+1}, ϕ and φ satisfy (3.19a)-(3.19d), *and the Riccati system* (3.17a)–(3.17n) *has no finite escape time.*

Proof. From the continuity condition (3.14k) on the Hamiltonian, we have for $i \in \mathcal{I}^n, n \in \mathcal{N}'$,

$$w_2 x(\tau^{*+}_{i,n}) + \lambda_2(\tau^{*+}_{i,n})(a x(\tau^{*+}_{i,n}) + b u(\tau^{*+}_{i,n})) = w_2 x(\tau^{*-}_{i,n}) + \lambda_2(\tau^{*-}_{i,n})(a x(\tau^{*-}_{i,n}) + b u(\tau^{*-}_{i,n})).$$

Using the continuity of the co-state of Player 2 (3.53), we can write the above equation as

$$w_2(x(\tau^{*+}_{i,n}) - x(\tau^{*-}_{i,n})) + a\lambda_2(\tau^*_{i,n})(x(\tau^{*+}_{i,n}) - x(\tau^{*-}_{i,n})) + \lambda_2(\tau^*_{i,n})b(u(\tau^{*+}_{i,n}) - u(\tau^{*-}_{i,n})) = 0.$$

On substituting $x(\tau^{*+}_{i,n}) - x(\tau^{*-}_{i,n}) = g v^*_{i,n}$, (3.43), and (3.47) in the above equation, we have

$$w_2 g v^*_{i,n} + a\lambda_2(\tau^*_{i,n}) g v^*_{i,n} - \frac{b^2}{c_u}\lambda_2(\tau^*_{i,n})(\lambda_1(\tau^{*+}_{i,n}) - \lambda_1(\tau^{*-}_{i,n})) = 0, \tag{3.21}$$

which on substituting (3.46) and (3.52) simplifies to

$$\begin{aligned} -w_2\frac{g^2}{c_v}\lambda_2(\tau^*_{i,n}) - a\frac{g^2}{c_v}\lambda_2(\tau^*_{i,n})^2 + \frac{b^2}{c_u}\lambda_2(\tau^*_{i,n})\left(z_1 x(\tau^*_{i,n}) + d_1\right) = 0, \\ \Rightarrow \lambda_2(\tau^*_{i,n})\left(\frac{c_u g^2}{b^2 c_v}(-w_2 - a\lambda_2(\tau^*_{i,n})) + z_1 x(\tau^*_{i,n}) + d_1\right) = 0. \end{aligned}$$

$\lambda_2(\tau^*_{i,n}) = 0$ implies that the equilibrium impulse level is zero. From Definition 3.1, $v^*_{i,n}$ cannot be equal to zero if $\tau^*_{i,n}$ is an admissible impulse instant. So, an impulse occurs if

$$x(\tau^*_{i,n}) = \frac{\frac{c_u g^2}{c_v b^2}(w_2 + a\lambda_2(\tau^*_{i,n})) - d_1}{z_1}.$$

We can rewrite (3.54) as

$$a\lambda_2(\tau^*_{i,n}) + w_2 = (a\lambda_2(t_{n+1}) + w_2)e^{a(t_{n+1}-\tau^*_{i,n})},$$

and substitute in the above equation to obtain

$$x(\tau^*_{i,n}) = \frac{\left(\frac{c_u g^2}{c_v b^2}(a\lambda_2(t_{n+1}) + w_2)e^{a(t_{n+1}-\tau^*_{i,n})}\right) - d_1}{z_1},\ i \in \mathcal{I}^n,\ n \in \mathcal{N}'. \tag{3.22}$$

On substituting (3.57) and (3.60a) in the above equation, we arrive at (3.20a). ■

Remark 3.5 *An impulse occurs at equilibrium whenever the state trajectory intersects the time varying function of gradient of the value function of Player 2, $\xi(t)$, given by*

$$\xi(t) = \frac{\left(\frac{c_u g^2}{c_v b^2}(aq_{2,n+1}(t_{n+1}) + w_2)e^{a(t_{n+1}-t)}\right) - d_1}{z_1}.$$

3.4.2 Non-linear optimization

Let $\tau_{1,n}, \tau_{2,n}, \cdots, \tau_{k_n,n}$ denote the admissible impulse instants for a given number of impulses, k_n, in each sampling interval $[t_n, t_{n+1}]$, $n \in \mathcal{N}'$. From Definition 3.1, we have

$$\tau_{1,n} < \tau_{2,n} < \cdots < \tau_{k_n,n}.$$

The above constraint can be represented as

$$D_n \boldsymbol{\tau}_n < \mathbf{0}, \tag{3.23}$$

where

$$D_n := \begin{bmatrix} 1 & -1 & 0 & \cdots & 0 & 0 \\ 0 & 1 & -1 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & -1 \end{bmatrix}_{(k_n-1)\times k_n}, \quad \boldsymbol{\tau}_n := \begin{bmatrix} \tau_{1,n} \\ \vdots \\ \tau_{k_n,n} \end{bmatrix}, \forall n \in \mathcal{N}'.$$

At the equilibrium impulse instants, the Hamiltonian continuity condition (3.20a) holds for the iLQDG formulated by (3.15). The equilibrium impulse instants are obtained by finding the fixed-point solution of the Riccati like system of equations (3.17a)–(3.17n) and the system of non-linear equality constraints

(3.20a). Alternatively, this problem can be viewed as the following constrained non-linear optimization problem:

$$\begin{aligned}
\underset{\{\boldsymbol{\tau}_n\}_{n\in\mathcal{N}'}}{\operatorname{argmin}} \quad & \sum_{n=1}^{N-1}\sum_{i=1}^{k_n}\left(x(\tau_{i,n})-\xi(\tau_{i,n})\right)^2 && && (3.24a)\\
\text{subject to} \quad & \mathbf{1}.(t_n+s)\leq \boldsymbol{\tau}_n \leq \mathbf{1}.(t_{n+1}-s) && \forall n\in\mathcal{N}' && (3.24b)\\
& D_n\boldsymbol{\tau}_n \leq -\mathbf{1}.s && \forall n\in\mathcal{N}', && (3.24c)
\end{aligned}$$

where $s>0$ is a slack variable, and

$$\xi(\tau_{i,n})=\frac{\left(\frac{c_u g^2}{c_v b^2}(aq_{2,n+1}(t_{n+1})+w_2)e^{a(t_{n+1}-\tau_{i,n})}\right)-d_1}{z_1}. \tag{3.24d}$$

The above problem can be solved using interior point algorithms (Byrd et al., 1999) or sequential quadratic programming methods (Büskens and Maurer, 2000).

3.5 Impulse linear-state differential game

In this section, we derive analytical expressions for the equilibrium number, timing and levels of impulses for the scalar linear-state differential game that can be obtained by setting $h_1=z_1=f_1=0$ in (3.15).

Theorem 3.4 *Let Assumptions 3.2 and 3.3 hold. Suppose that $t_1,t_2,\ldots,t_N$ are the sampling instants. Then, there can be at most one impulse in the sampled-data Nash equilibrium with the timing and level of impulse given by*

$$\tau^* = T-\frac{1}{a}\ln\left(\frac{b^2c_v}{g^2c_u}\frac{d_1}{as_2+w_2}\right), \tag{3.25a}$$

$$v=\frac{gw_2}{c_v a}-\frac{b^2 d_1}{agc_u}. \tag{3.25b}$$

Further, τ^* is the equilibrium impulse instant if the following condition holds for some $n\in\mathcal{N}'$:

$$t_n < T-\frac{1}{a}\ln\left(\frac{b^2c_v}{g^2c_u}\frac{d_1}{as_2+w_2}\right) < t_{n+1}. \tag{3.25c}$$

The equilibrium control of Player 1 is given by

$$u^*(t)=-\frac{b}{c_u}\lambda_1(t), \tag{3.25d}$$

where

$$\dot{\lambda}_1(t) = -a\lambda_1(t) - w_1,\ \lambda_1(t_{n+1}) = q_{1,n+1}(t_{n+1}),\ n \in \mathcal{N}',$$
$$\dot{q}_{1,n}(t) = -w_1 - aq_{1,n}(t),\ q_{1,n}(t_{n+1}) = q_{1,n+1}(t_{n+1}), n \in \mathcal{N}',\ q_{1,N}(T) = s_1,$$
$$q_{1,n}(\tau^{*-}) = q_{1,n}(\tau^{*+}) + d_1.$$

Proof. The Hamiltonian of Player 1 is given by

$$H_1(x(t), u(t), \lambda_1(t)) := w_1 x(t) + \frac{1}{2}c_u u(t)^2 + \lambda_1(t)(ax(t) + bu(t)).$$

Using (3.14a) and Assumption 3.3 on interior solutions, the first-order condition yields

$$H_{1u}(x^*(t), u^*(t), \lambda_1(t)) = 0 \Rightarrow u^*(t) = -\frac{b}{c_u}\lambda_1(t). \tag{3.26}$$

From (3.14e), (3.14f), and (3.14i), the equilibrium state and co-state trajectories during the non-impulse instants evolves as follows:

$$\dot{x}^*(t) = ax^*(t) - \frac{b^2}{c_u}\lambda_1(t),\ x^*(t_n) = x_n,\ n \in \mathcal{N}', \tag{3.27}$$
$$\dot{\lambda}_1(t) = -a\lambda_1(t) - w_1,\ \lambda_1(t_{n+1}) = \frac{\partial V_1^*(t_{n+1}, x(t_{n+1}))}{\partial x},\ n \in \mathcal{N}', \tag{3.28}$$

and at the impulse instants, the co-state jumps according to

$$\lambda_1(\tau_{i,n}^{*-}) = \lambda_1(\tau_{i,n}^{*+}) + d_1,\ i \in \mathcal{I}^n, n \in \mathcal{N}'. \tag{3.29}$$

Using the approach in Theorem 3.2, it can be shown that the equilibrium value-to-go of Player 1 is given by $V_1^*(t_n, x_n) = q_{1,n}(t_n)x_n + r_{1,n}(t_n)$, $\forall n \in \mathcal{N}'$, such that

$$\dot{q}_{1,n}(t) = -w_1 - aq_{1,n}(t),\ q_{1,n}(t_{n+1}) = q_{1,n+1}(t_{n+1}), n \in \mathcal{N}',\ q_{1,N}(T) = s_1, \tag{3.30a}$$
$$\dot{r}_{1,n}(t) = -\left(\frac{\lambda_1(t)}{2} - q_{1,n}(t)\right)\frac{b^2}{c_u}\lambda_1(t),$$
$$r_{1,n}(t_{n+1}) = r_{1,n+1}(t_{n+1}),\ r_{1,N}(T) = 0, \tag{3.30b}$$
$$q_{1,n}(\tau_{i+1,n}^{*-}) = q_{1,n}(\tau_{i+1,n}^{*+}) + d_1,\ i \in \mathcal{I}^n, \tag{3.30c}$$
$$r_{1,n}(\tau_{i+1,n}^{*-}) = r_{1,n}(\tau_{i+1,n}^{*+}) + r_{1,n}(\tau_{i+1,n}^{*+})gv_{i,n}^*,\ i \in \mathcal{I}^n. \tag{3.30d}$$

From (3.28), we obtain

$$\lambda_1(t_{n+1}) = q_{1,n+1}(t_{n+1}), \forall n \in \mathcal{N}'. \tag{3.31}$$

Given the equilibrium control $u^*(.)$ of Player 1, the necessary optimality conditions for Player 2 are given by (3.14b), (3.14g), (3.14h), (3.14j), (3.14k). The co-state of Player 2 evolves according to the following equation

$$\lambda_2(t) = -a\lambda_2(t) - w_2,\ \forall t \notin \mathcal{T}^n,\ n \in \mathcal{N}', \lambda_2(T) = s_2, \tag{3.32}$$

$$\lambda_2(\tau_{i,n}^{*+}) = \lambda_2(\tau_{i,n}^{*-}),\ i \in \mathcal{I}^n,\ n \in \mathcal{N}'. \tag{3.33}$$

The jump in the state at the impulse instant is given

$$x(\tau_{i,n}^{*+}) = x(\tau_{i,n}^{*-}) - \frac{g^2}{c_v}\lambda_2(\tau_{i,n}^{*+}),\ i \in \mathcal{I}^n,\ n \in \mathcal{N}'. \tag{3.34}$$

From the proof of Theorem 3.2, we can show that the value-to-go for Player 2 at any time t is given by $V_2(t_n, x_n) = q_{2,n}(t_n)x_n + r_{2,n}(t_n)$ such that, for $t \notin \mathcal{T}^n,\ n \in \mathcal{N}'$, we have

$$\dot{q}_{2,n}(t) = -w_2 - aq_{2,n}(t),\ q_{2,n}(t_{n+1}) = q_{2,n+1}(t_{n+1}),\ q_{2,N}(T) = s_2, \tag{3.35a}$$

$$\dot{r}_{2,n}(t) = q_{2,n}(t)\frac{b^2}{c_u}\lambda_1(t),\ r_{2,n}(t_{n+1}) = r_{2,n+1}(t_{n+1}),\ r_{2,N}(T) = 0, \tag{3.35b}$$

and for $i \in \mathcal{I}^n,\ n \in \mathcal{N}'$, we have

$$q_{2,n}(\tau_{i,n}^{*-}) = q_{2,n}(\tau_{i,n}^{*+}), \tag{3.35c}$$

$$r_{2,n}(\tau_{i+1,n}^{*-}) = r_{2,n}(\tau_{i+1,n}^{*+}) + \frac{g^2}{2c_v}\lambda_2(\tau_{i+1,n}^{*+})^2 - q_{2,n}(\tau_{i+1,n}^{*+})\left(\frac{g^2}{c_v}\lambda_2(\tau_{i+1,n}^{*+})\right). \tag{3.35d}$$

From the above equations for the evolution of q_2, and (3.32), (3.33), we can see that $q_2(.)$ and $\lambda_2(.)$ have the same dynamics and terminal conditions in each sampling interval, and are continuous functions of time, and thus we obtain

$$\lambda_2(t) = q_2(t) = -\frac{w_2}{a} + (s_2 + \frac{w_2}{a})e^{a(T-t)},\ \forall t \in [0, T]. \tag{3.36}$$

At the impulse instants $\tau_{i,n}^*$, the Hamiltonian continuity condition (3.14k) holds, which implies

$$w_2x(\tau_{i,n}^{*+}) + \lambda_2(\tau_{i,n}^{*+})(ax(\tau_{i,n}^{*+}) + bu(\tau_{i,n}^{*+})) = w_2x(\tau_{i,n}^{*-}) + \lambda_2(\tau_{i,n}^{*-})(ax(\tau_{i,n}^{*-}) + bu(\tau_{i,n}^{*-})).$$

Using the conditions (3.29), (3.33), (3.34), we can rewrite the Hamiltonian continuity condition as

$$-\left((w_2 + a\lambda_2(\tau_{i,n}^*))\frac{g^2}{c_v} - \frac{b^2}{c_u}d_1\right)\lambda_2(\tau_{i,n}^*) = 0,$$

This implies that if an impulse occurs at $\tau_{i,n}^*$, then $\lambda(\tau_{i,n}^*)$ can take the following two values:

$$\lambda_2(\tau_{i,n}^*) = 0, \frac{b^2 c_v}{g^2 c_u}\frac{d_1}{a} - \frac{w_2}{a}.$$

$\lambda_2(\tau_{i,n}^*) = 0$ implies that the impulse level is zero. The admissible impulse instants in Definition 3.1 are such that if $\tau_{i,n}^*$ is an equilibrium impulse instant, then the impulse level is not equal to 0. Since $\lambda_2(t)$ is strictly monotone for $t \in [0, T]$, we obtain a unique solution:

$$\lambda_2(\tau_{i,n}^*) = \frac{b^2 c_v}{g^2 c_u}\frac{d_1}{a} - \frac{w_2}{a}. \tag{3.37}$$

From (3.36) and (3.37), we obtain a unique equilibrium impulse instant

$$\tau^* = T - \frac{1}{a}\ln\left(\frac{b^2 c_v}{g^2 c_u}\frac{d_1}{a s_2 + w_2}\right).$$

For τ^* to be an interior impulse, we must have for some $n \in \mathcal{N}'$,

$$t_n < \tau^* < t_{n+1} \Rightarrow t_n < T - \frac{1}{a}\ln\left(\frac{b^2 c_v}{g^2 c_u}\frac{d_1}{a s_2 + w_2}\right) < t_{n+1}.$$

The equilibrium impulse level is given by

$$v^* = -\frac{g}{c_v}\lambda_2(\tau^*) = -\frac{g}{c_v}\left(\frac{b^2 c_v}{g^2 c_u}\frac{d_1}{a} - \frac{w_2}{a}\right). \tag{3.38}$$

■

Clearly, there can be at most one impulse in the sampled-data Nash equilibrium of our specialized scalar linear-state differential game with impulse controls. Next, we consider a variation of the game where the problem parameters of Player 1 and Player 2 vary with time, and are constant between the sampling instants:

$$\begin{aligned}
J_1(x_0, u(\cdot), \tilde{v}) &= \sum_{n=1}^{N-1}\Big(\int_{t_n}^{t_{n+1}}\left(w_{1,n}x(t) + \frac{1}{2}c_{u,n}u(t)^2\right)dt + \sum_{i=1}^{k_n} d_{1,n}x(\tau_{i,n}^-)\Big) + s_1 x(T),\\
J_2(x_0, u(\cdot), \tilde{v}) &= \sum_{n=1}^{N-1}\Big(\int_{t_n}^{t_{n+1}} w_{2,n}x(t)dt + \sum_{i=1}^{k_n}\left(\frac{1}{2}c_{v.n}v_{i,n}^2\right)\Big) + s_2 x(T),\\
\dot{x}(t) &= a_n x(t) + b_n u(t),\ t \neq \mathcal{T}^n,\ x(0) = x_0,\\
x(\tau_{i,n}^+) &= x(\tau_{i,n}^-) + g_n v_{i,n},\ i \in \mathcal{I}^n,
\end{aligned} \tag{3.39}$$

where the state at the sampling instants $t_1, t_2, \cdots, t_N$ is denoted by $x_1, x_2, \cdots, x_N$.

Theorem 3.5 *Let Assumptions 3.2 and 3.3 hold. Suppose that $t_1, t_2, \ldots, t_N$ are the sampling instants. Then, there can be at most one impulse in each sampling interval, and at most N impulses in the sampled-data Nash equilibria with the timing and level of impulses given by*

$$\tau_n^* = t_{n+1} - \frac{1}{a_n} \ln\left(\frac{b_n^2 c_{v,n}}{g_n^2 c_{u,n}} \frac{d_{1,n}}{a_n \lambda_2(t_{n+1}) + w_{2,n}}\right), \tag{3.40a}$$

$$v_n^* = \frac{g_n w_{2,n}}{c_{v,n} a_n} - \frac{b_n^2 d_{1,n}}{a_n g_n c_{u,n}}, \tag{3.40b}$$

where

$$\lambda_2(t) = -\frac{w_{2,n}}{a_n} + (\lambda_2(t_{n+1}) + \frac{w_{2,n}}{a_n})e^{a_n(t_{n+1}-t)}, \forall t \in [t_n, t_n+1),\ n \in \mathcal{N}', \tag{3.40c}$$
$$\lambda_2(T) = s_2.$$

Further, τ_n^* is an equilibrium impulse instant if the following conditions hold:

$$t_n < t_{n+1} - \frac{1}{a_n} \ln\left(\frac{b_n^2 c_{v,n}}{g_n^2 c_{u,n}} \frac{d_{1,n}}{a_n \lambda_2(t_{n+1}) + w_{2,n}}\right) < t_{n+1},\ n \in \mathcal{N}'.$$

The equilibrium strategy of Player 1 is given by

$$u^*(t) = -\frac{b_n}{c_{u,n}} \lambda_1(t), \tag{3.40d}$$

where, for $n \in \mathcal{N}'$,

$$\dot{\lambda}_1(t) = -a_n \lambda_1(t) - w_{1,n},\ \lambda_1(t_{n+1}) = q_{1,n+1}(t_{n+1}),$$
$$\dot{q}_{1,n}(t) = -w_{1,n} - a_n q_{1,n}(t),\ q_{1,n}(t_{n+1}) = q_{1,n+1}(t_{n+1}),\ q_{1,N}(T) = s_1,$$
$$q_{1,n}(\tau_n^{*-}) = q_{1,n}(\tau_n^{*+}) + d_{1,n}.$$

Proof. Using the proof of Theorem 3.4, we obtain

$$\lambda_2(t) = -\frac{w_{2,n}}{a_n} + (\lambda_2(t_{n+1}) + \frac{w_{2,n}}{a_n})e^{a_n(t_{n+1}-t)}, \forall t \in [t_n, t_n+1),\ n \in \mathcal{N}', \tag{3.41}$$
$$\lambda_2(T) = s_2.$$

Between the sampling instants t_n and t_{n+1}, the Hamiltonian continuity condition holds at the impulse instants, which implies

$$-\left((w_{2,n} + a_n \lambda_2(\tau_{i,n}^*)) \frac{g_n^2}{c_{v,n}} - \frac{b_n^2}{c_{u,n}} d_{1,n}\right) \lambda_2(\tau_{i,n}^*) = 0,$$

From the continuity and strict monotonicity of co-state in each sampling interval, we obtain a unique value of co-state in each sampling interval

$$\lambda_2(\tau_n^*) = \frac{b_n^2 c_{v,n}}{g_n^2 c_{u,n}} \frac{d_{1,n}}{a_n} - \frac{w_{2,n}}{a_n}. \tag{3.42}$$

Substituting (3.42) in (3.41), we obtain

$$\tau_n^* = t_{n+1} - \frac{1}{a_n} \ln \left(\frac{b_n^2 c_{v,n}}{g_n^2 c_{u,n}} \frac{d_{1,n}}{a_n \lambda_2(t_{n+1}) + w_{2,n}} \right).$$

The equilibrium impulse level is then given by

$$v_n^* = -\frac{g_n}{c_{v,n}} \lambda_2(\tau_n^*) = \frac{g_n w_{2,n}}{c_{v,n} a_n} - \frac{b_n^2 d_{1,n}}{a_n g_n c_{u,n}}.$$

From the proof of Theorem 3.2, we also obtain the equilibrium controls of Player 1 by replacing the problem parameters in each sampling interval by the time-varying parameters. ■

3.6 A numerical example

In this section, we illustrate the theory developed in the previous sections using a numerical example.

Consider a dynamic game where Player 1's profit is decreasing quadratically with the state while Player 2's profit increases linearly with the state. The time horizon of the game is $T = 20$. Player 1 uses piecewise continuous sampled-data state feedback controls while Player 2 uses impulse controls. The state measurements are made at given instants of time $t_1 = 0$, $t_2 = 10$, $t_3 = 20$. Player 1 and Player 2 maximize their respective objective functions

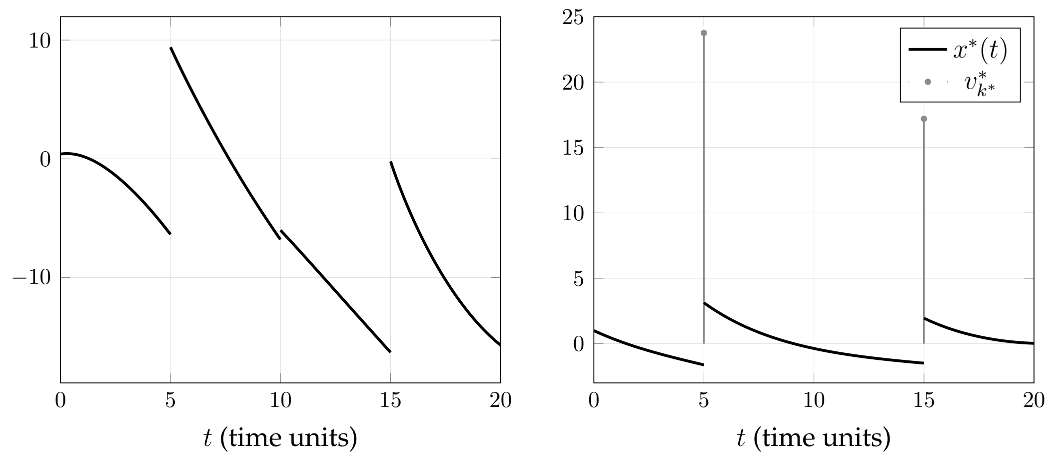
$$\begin{aligned} J_1(x_0, u(\cdot), \tilde{v}) = \sum_{n=1}^{2} \Big(\int_{t_n}^{t_{n+1}} \left(-x(t)^2 - 4x(t) - 3u(t)^2\right) dt - 4x(\tau_n^-)^2 \Big) \\ - 2x(20)(x(20) + 1) \\ J_2(x_0, u(\cdot), \tilde{v}) = \sum_{n=1}^{2} \Big(\int_{t_n}^{t_{n+1}} 10x(t)dt - 0.25v_n^2 \Big) + 6x(20), \end{aligned}$$

and the state dynamics are given by

$$\dot{x}(t) = -0.1x(t) + 0.4u(t),\ t \notin \{\tau_1, \tau_2\},\ x(0) = 1,$$

$$x(\tau_i^+) = x(\tau_i^-) + 0.2v_i,\ i \in \{1, 2\}.$$

First, we analyze the case where the impulses are periodic, that is, $\tau_1 = 5$ and $\tau_2 = 15$. The equilibrium control of Player 1, given in Figure 3.1a, jumps at the impulse instants because of the jump in her co-state caused by the impulse control of Player 2. The state trajectory, and the equilibrium impulse levels of Player 2 are shown in Figure 3.1b. At equilibrium, Player 1 incurs a loss of 238.37, while Player 2 incurs a loss of 203.09.



(a) $u^*(t)$ **(b)** $x^*(t)$ and v^*

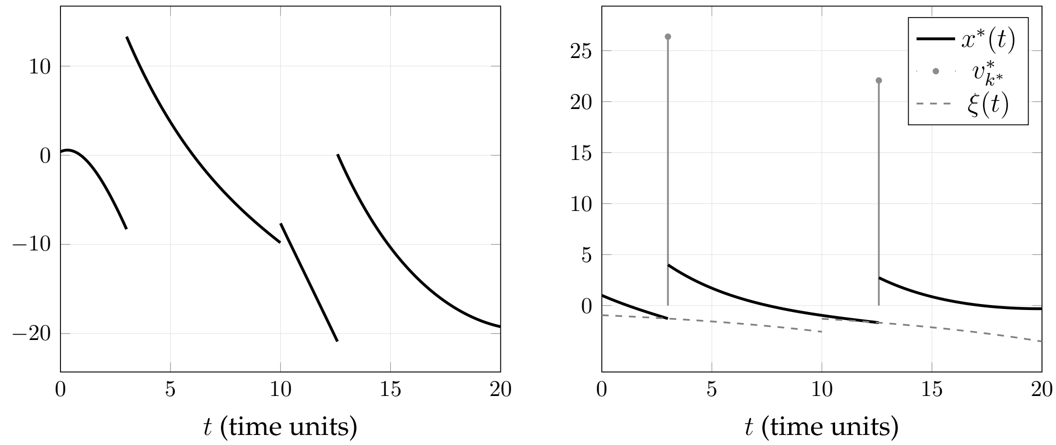
Figure 3.1 – Equilibrium controls, and state trajectory with periodic impulses.

Next, we determine the equilibrium when the impulse instants in each sampling interval are determined by Player 2, and there is one impulse in each sampling interval. The impulse timing is characterized by the Hamiltonian continuity condition (3.21) which reduces to determining the time at which state trajectory intersects $\xi(t)$ as shown in Figure 3.2b

$$\xi(t) = \begin{cases} -3.52e^{-0.1(10-t)} & t \in [0, 10) \\ -2.57e^{-0.1(20-t)} & t \in (10, 20] \end{cases}.$$

The equilibrium impulses occur at $\tau_1^* = 3$ and $\tau_2^* = 12.59$, and at equilibrium, the losses of Player 1 and Player 2 are given by 311.64 and 232.83, respectively. The piecewise-continuous equilibrium control of Player 1 is shown in Figure 3.2a and equilibrium impulse levels of Player 2 are shown in Figure 3.2b.

Clearly, both players incur higher loss if Player 2 determines the timing of impulses when compared with the case where impulse timings are periodic. This illustrates a well-known result that enlarging the strategy space of a player does not necessarily benefit the player in a game problem.



(a) $u^*(t)$ **(b)** $x^*(t)$, $v^*_{k^*}$ and $\xi(t)$

Figure 3.2 – Equilibrium controls, and state and co-state trajectories.

3.7 Conclusions

In this paper, we have derived necessary conditions for the existence of sampled-data Nash equilibrium in a general class of two-player nonzero-sum differential games with impulse controls, where only one of the players controls the impulses (their number, timing and magnitudes). For a scalar linear-quadratic differential game, we have shown that the sampled-data Nash equilibrium can be obtained by determining the fixed point of a system of Riccati like equations with jumps coupled with non-linear equality constraints. We have also shown that the equilibrium piecewise-continuous control of Player 1 is linear in the most recently measured state value, and provide a numerical procedure to determine the equilibrium. Further, we have shown that there can be at most one impulse in the sampled-data Nash equilibrium of a scalar linear-state differential game with impulse controls, and for the case with time-varying parameters, there can be at most one impulse in each sampling interval, and we have obtained analytical expressions for equilibrium timing and level of impulses.

For the future, it would be interesting to apply our results to case studies in pollution regulation, exchange rate interventions, and cybersecurity. One extension of our work would be to differential games where both players use continuous as well as impulse controls. Another extension would be to differential games with more than two players.

3.8 Appendix

3.8.1 Proof of Theorem 3.2

Given the equilibrium control of Player 2, we obtain necessary conditions for iLQDG using (3.14a), (3.14e), (3.14f), (3.14i). The Hamiltonian of Player 1 is given by

$$H_1(x(t), u(t), \lambda_1(t)) := \frac{1}{2}h_1 x(t)^2 + w_1 x(t) + \frac{1}{2}c_u u(t)^2 + \lambda_1(t)(ax(t) + bu(t)),$$

where $\lambda_1(t)$ is the co-state of Player 1. Using (3.14a) and Assumption 3.3 on interior solutions, the first-order condition yields

$$H_{1u}(x^*(t), u^*(t), \lambda_1(t)) = 0 \Rightarrow u^*(t) = -\frac{b}{c_u}\lambda_1(t). \tag{3.43}$$

From (3.14e) and (3.14f), the equilibrium state and co-state trajectory at the non-impulse instants evolve as follows:

$$\dot{x}^*(t) = ax^*(t) - \frac{b^2}{c_u}\lambda_1(t),\ x^*(t_{n+1}) = x_{n+1}, \tag{3.44}$$

$$\dot{\lambda}_1(t) = -a\lambda_1(t) - h_1 x^*(t) - w_1,\ \lambda_1(t_{n+1}) = \frac{\partial V_1^*(t_{n+1}, x(t_{n+1}))}{\partial x}. \tag{3.45}$$

From (3.14i), the jump in the co-state at the impulse instants is given by

$$\lambda_1(\tau_{i,n}^{*-}) = \lambda_1(\tau_{i,n}^{*+}) + z_1 x^*(\tau_{i,n}^{*-}) + d_1. \tag{3.46}$$

Given that the objective of Player 1 is quadratic in state, we can guess the form of co-state to be linear in state so that

$$\lambda_1(t) = \alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t),\ \forall t \in [t_n, t_{n+1}),\ n \in \mathcal{N}'. \tag{3.47}$$

We substitute (3.47) in (3.46) to obtain the following relation at the impulse instants:

$$\alpha_{1,n}(\tau_{i,n}^{*-})x^*(\tau_{i,n}^{*-}) + \beta_{1,n}(\tau_{i,n}^{*-}) = \alpha_{1,n}(\tau_{i,n}^{*+})x^*(\tau_{i,n}^{*+}) + \beta_{1,n}(\tau_{i,n}^{*+}) + z_1 x^*(\tau_{i,n}^{*-}) + d_1$$

$$= \alpha_{1,n}(\tau_{i,n}^{*+})(x^*(\tau_{i,n}^{*-}) + gv_{i,n}^*) + \beta_{1,n}(\tau_{i,n}^{*+}) + z_1 x^*(\tau_{i,n}^{*-}) + d_1,$$

where $v_{i,n}^*$ denotes the equilibrium impulse level of Player 2 at the impulse instant $\tau_{i,n}^*$. On comparing the coefficients, we obtain

$$\alpha_{1,n}(\tau_{i,n}^{*-}) = \alpha_{1,n}(\tau_{i,n}^{*+}) + z_1,\ \forall i \in \mathcal{I}^n,\ n \in \mathcal{N}',$$
$$\beta_{1,n}(\tau_{i,n}^{*-}) = \beta_{1,n}(\tau_{i,n}^{*+}) + \alpha_{1,n}(\tau_{i,n}^{*+}) g v_{i,n}^* + d_1,\ \forall i \in \mathcal{I}^n,\ n \in \mathcal{N}'.$$

Taking the derivative of (3.47) with respect to time, we obtain

$$\dot{\lambda}_1(t) = \dot{\alpha}_{1,n}(t) x^*(t) + \alpha_{1,n}(t)\dot{x}^*(t) + \dot{\beta}_{1,n}(t).$$

Using the derivatives of state and co-state from (3.44) and (3.45) in the above equation, we get

$$-a\lambda_1(t) - h_1 x^*(t) - w_1 = \dot{\alpha}_{1,n}(t)x^*(t) + \alpha_{1,n}(t)\left(ax^*(t) - \frac{b^2}{c_u}\lambda_1(t)\right) + \dot{\beta}_{1,n}(t).$$

Substitute (3.47) in the above equation to obtain

$$\begin{aligned} &- a(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t)) - h_1 x^*(t) - w_1 \\ &\quad = \dot{\alpha}_{1,n}(t)x^*(t) + \alpha_{1,n}(t)\left(ax^*(t) - \frac{b^2}{c_u}(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t))\right) + \dot{\beta}_{1,n}(t). \end{aligned}$$

On comparing the coefficients, we obtain

$$\dot{\alpha}_{1,n}(t) = -2\alpha_{1,n}(t)a + \frac{b^2}{c_u}\alpha_{1,n}(t)^2 - h_1,\ \forall t \notin \mathcal{T}^n, n \in \mathcal{N},\ \alpha_{1,n}(T) = f_1,$$
$$\dot{\beta}_{1,n}(t) = \beta_{1,n}(t)\left(\frac{b^2}{c_u}\alpha_{1,n}(t) - a\right) - w_1,\ \forall t \notin \mathcal{T}^n,\ n \in \mathcal{N},\ \beta_{1,n}(T) = s_1,$$

where $\alpha_{1,n}(t_{n+1})x(t_{n+1}) + \beta_{1,n}(t_{n+1}) = \frac{\partial V_1^*(t_{n+1}, x_{n+1})}{\partial x}$. The value-to-go for Player 1 is given by

$$\begin{aligned} V_1(t_n, x_n) = \sum_{i=1}^{k_n^*} \Bigg(\frac{1}{2}\left(\int_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} \left(h_1 x(t)^2 + 2w_1 x(t) + c_u u(t)^2\right) dt\right) \\ + \frac{1}{2} z_1 x(\tau_{i,n}^{*-})^2 + d_1 x(\tau_{i,n}^{*-})\Bigg) + V_1(t_{n+1}, x_{n+1}), \end{aligned} \tag{3.48}$$

where $\tau_{k_n^*+1}^* := t_{n+1}$. Next, we know that for all x,

$$\int_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} \left(\frac{1}{2}\dot{p}_{1,n}(t)x(t)^2 + p_{1,n}(t)x(t)\dot{x}(t) + \dot{q}_{1,n}(t)x(t) + q_{1,n}(t)\dot{x}(t) + \dot{r}_{1,n}(t)\right) dt$$

$$- \frac{1}{2}p_{1,n}(t)x(t)^2\Big|_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} - q_{1,n}(t)x(t)\Big|_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} - r_{1,n}(t)\Big|_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} = 0, i \in \mathcal{I}^n,\ n \in \mathcal{N}'.$$

Substituting $\dot{x}(t) = ax(t) + bu(t)$ in the above equation and adding it to (3.48) gives

$$\begin{aligned}
V_1(t_n, x_n) =& \sum_{i=1}^{k_n^*} \Big(\int_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} \Big(\frac{1}{2}c_u u(t)^2 + p_{1,n}(t)x(t)bu(t) + q_{1,n}(t)bu(t)\Big)\, dt \\
&+ \int_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} \Big(\frac{1}{2}h_1 x(t)^2 + w_1 x(t) + \frac{1}{2}\dot{p}_{1,n}(t)x(t)^2 + p_{1,n}(t)ax(t)^2 + \dot{q}_{1,n}(t)x(t) \\
&\quad + q_{1,n}(t)ax(t) + \dot{r}_{1,n}(t)\Big)dt - \frac{1}{2}p_{1,n}(t)x(t)^2\Big|_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} - q_{1,n}(t)x(t)\Big|_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} \\
&\quad - r_{1,n}(t)\Big|_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} + \frac{1}{2}z_1 x(\tau^{*-}_{i,n})^2 + d_1 x(\tau^{*-}_{i,n})\Big) + V_1(t_{n+1}, x_{n+1}).
\end{aligned}$$

Substituting the equilibrium control $u^*(t) = -\frac{b}{c_u}(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t))$, we obtain

$$\begin{aligned}
V_1^*(t_n, x_n) =& \sum_{i=1}^{k_n} \Big(\int_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} \Big(\frac{1}{2}c_u(-\frac{b}{c_u}(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t)))^2 - (p_{1,n}(t)x^*(t) \\
&+ q_{1,n}(t))\frac{b^2}{c_u}(\alpha_{1,n}(t)x^*(t) + \beta_{1,n}(t))\Big)dt \quad + \int_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} \big(\frac{h_1 x^*(t)^2}{2} \\
&+ w_1 x^*(t) + \frac{1}{2}\dot{p}_{1,n}(t)x^*(t)^2 + p_{1,n}(t)ax^*(t)^2 + \dot{q}_{1,n}(t)x^*(t) + q_{1,n}(t)ax^*(t) \\
&+ \dot{r}_{1,n}(t)\big)dt - \frac{1}{2}p_{1,n}(t)x^*(t)^2\Big|_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} - q_{1,n}(t)x^*(t)\Big|_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} - r_{1,n}(t)\Big|_{\tau^{*+}_{i,n}}^{\tau^{*-}_{i+1,n}} \\
&+ \frac{1}{2}z_1 x^*(\tau^{*-}_{i,n})^2 + d_1 x^*(\tau^{*-}_{i,n})\Big) + V_1^*(t_{n+1}, x_{n+1}).
\end{aligned}$$

Since the equilibrium control maximizes the value-to-go for Player 1, the following relations hold for all $n \in \mathcal{N}'$:

$$\begin{aligned}
&\dot{p}_{1,n}(t) = -h_1 - 2(a - \frac{b^2}{c_u}\alpha_{1,n}(t))p_{1,n}(t) - \frac{b^2}{c_u}\alpha_{1,n}(t)^2,\ \forall t \notin \mathcal{T}^n, \\
&\dot{q}_{1,n}(t) = -w_1 + \frac{b^2}{c_u}p_{1,n}(t)\beta_{1,n}(t) - q_{1,n}(t)(a - \frac{b^2}{c_u}\alpha_{1,n}(t)) - \frac{b^2}{c_u}\alpha_{1,n}(t)\beta_{1,n}(t),\ \forall t \notin \mathcal{T}^n, \\
&\dot{r}_{1,n}(t) = \frac{b^2}{c_u}\left(q_{1,n}(t)\beta_{1,n}(t) - \beta_{1,n}(t)^2\right),\ \forall t \notin \mathcal{T}^n, \\
&p_{1,n}(\tau^{*-}_{i+1,n}) = p_{1,n}(\tau^{*+}_{i+1,n}) + z_1,\ \forall i \in \mathcal{I}^n, \\
&q_{1,n}(\tau^{*-}_{i+1,n}) = p_{1,n}(\tau^{*+}_{i+1,n})gv^*_{i,n} + q_{1,n}(\tau^{*+}_{i+1,n}) + d_1,\ \forall i \in \mathcal{I}^n, \\
&r_{1,n}(\tau^{*-}_{i+1,n}) = r_{1,n}(\tau^{*+}_{i+1,n}) + \frac{1}{2}p_{1,n}(\tau^{*+}_{i+1,n})g^2 v^{*2}_{i,n} + q_{1,n}(\tau^{*+}_{i+1,n})gv^*_{i,n},\ \forall i \in \mathcal{I}^n,
\end{aligned}$$

$$p_{1,n}(t_{n+1}) = p_{1,n+1}(t_{n+1}),\ q_{1,n}(t_{n+1}) = q_{1,n+1}(t_{n+1}),\ r_{1,n}(t_{n+1}) = r_{1,n+1}(t_{n+1}),$$

where the last set of equations hold for $n \in \mathcal{N}'$ because there are no impulses at the sampling instants (see Definition 3.1). Therefore, the equilibrium value-to-go is given by

$$V_1^*(t_n, x_n) = \frac{1}{2}p_{1,n}(t_n)x_n^2 + q_{1,n}(t_n)x_n + r_{1,n}(t_n),\ \forall n \in \mathcal{N}'. \tag{3.49}$$

The Hamiltonian, and the impulse Hamiltonian of Player 2 are given by

$$\begin{aligned} H_2(x(t), u(t), \lambda_2(t)) &:= w_2 x(t) + \lambda_2(t)(ax(t) + bu(t)), \\ H_2^I(v_i, \lambda_2(\tau_i^+)) &:= \frac{1}{2}c_v v_i^2 + \lambda_2(\tau_{i,n}^+)gv_i, \end{aligned}$$

where $\lambda_2(t)$ is the co-state of Player 2. From (3.14g), we obtain the dynamics of the co-state of Player 2 at the non-impulse instants as follows:

$$\dot{\lambda}_2(t) = -a\lambda_2(t) - w_2,\ \forall t \in (t_n, t_{n+1}),\ n \in \mathcal{N},\ \lambda_2(t_{n+1}) = \frac{\partial V_2^*(t_{n+1}, x_{n+1})}{\partial x}. \tag{3.50}$$

The co-state is equal to the gradient of the value function of Player 2 at the sampling instants because of our assumption that there are no impulses at the sampling instants. Using the necessary condition (3.14b) and Assumption 3.3 on interior impulse levels, the first-order condition yields

$$H_{1v_i}(v_{i,n}^*, \lambda_2(\tau_{i,n}^*)) = 0 \Rightarrow v_{i,n}^* = -\frac{g}{c_v}\lambda_2(\tau_{i,n}^{*+}),\ \forall i \in \mathcal{I}^n, n \in \mathcal{N}'. \tag{3.51}$$

Since $v_{i,n}^*$ are the equilibrium impulse levels, it follows from (3.14h) that the jump in the state is given by

$$x(\tau_{i,n}^{*+}) = x(\tau_{i,n}^{*-}) - \frac{g^2}{c_v}\lambda_2(\tau_{i,n}^{*+}),\ \forall i \in \mathcal{I}^n, n \in \mathcal{N}', \tag{3.52}$$

and from (3.14j), we have that the co-state of Player 2 is continuous, that is

$$\lambda_2(\tau_{i,n}^{*-}) = \lambda_2(\tau_{i,n}^{*+}),\ \forall i \in \mathcal{I}^n, n \in \mathcal{N}. \tag{3.53}$$

Also, from the continuity of co-state at the impulse instants and (3.50), we obtain

$$\begin{aligned} \lambda_2(t) = &-\frac{w_2}{a} + (\lambda_2(t_{n+1}) + \frac{w_2}{a})e^{a(t_{n+1}-t)},\ \forall t \in [t_n, t_{n+1}), \\ &\lambda_2(t_{n+1}) = \frac{\partial V_2^*(t_{n+1}, x_{n+1})}{\partial x},\ n \in \mathcal{N}', \lambda_2(T) = s_2. \end{aligned} \tag{3.54}$$

The value-to-go for Player 2 is given by

$$V_2(t_n, x_n) = \sum_{i=1}^{k_n} \left(\int_{\tau_{i,n}^+}^{\tau_{i+1,n}^-} w_2 x(t) dt + \frac{1}{2} c_v v_{i,n}^2 \right) + V_2(t_{n+1}, x_{n+1}). \tag{3.55}$$

For all x, we have

$$\int_{\tau_{i,n}^+}^{\tau_{i+1,n}^-} \left(\dot{q}_{2,n}(t)x(t) + q_{2,n}(t)\dot{x}(t) + \dot{r}_{2,n}(t) \right) dt - q_{2,n}(t)x(t)\Big|_{\tau_{i,n}^+}^{\tau_{i+1,n}^-} \\ - r_{2,n}(t)\Big|_{\tau_{i,n}^+}^{\tau_{i+1,n}^-} = 0, \; \forall i \in \mathcal{I}^n, \; n \in \mathcal{N}'.$$

Substituting $\dot{x}(t) = ax(t) + bu^*(t)$ in the above equation and adding it to (3.55) yields

$$V_2(t_n, x_n) = \sum_{i=1}^{k_n} \Big(\int_{\tau_{i,n}^+}^{\tau_{i+1,n}^-} \left(w_2 x(t) + \dot{q}_{2,n}(t)x(t) + q_{2,n}(t)ax(t) + q_{2,n}(t)bu^*(t) + \dot{r}_{2,n}(t) \right) dt \\ - q_{2,n}(t)x(t)\Big|_{\tau_{i,n}^+}^{\tau_{i+1,n}^-} - r_{2,n}(t)\Big|_{\tau_{i,n}^-}^{\tau_{i+1,n}^+} + \frac{1}{2} c_v v_{i,n}^2 \Big) + V_2(t_{n+1}, x_{n+1}).$$

On substituting the equilibrium controls $(\tau_{i,n}^*, v_{i,n}^*)$, $i \in \mathcal{I}^n$, $n \in \mathcal{N}'$, we obtain the equilibrium value-to-go $V_2^*(t_n, x_n)$, so that

$$V_2^*(t_n, x_n) \\ = \sum_{i=1}^{k_n^*} \Big(\int_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} \Big((w_2 + \dot{q}_{2,n}(t) + q_{2,n}(t)(a - \frac{b^2}{c_u}\alpha_{1,n}(t)))x^*(t) + \dot{r}_{2,n}(t) \\ - q_{2,n}(t)\frac{b^2}{c_u}\beta_{1,n}(t) \Big) dt - q_{2,n}(t)x^*(t)\Big|_{\tau_{i,n}^{*+}}^{\tau_{i+1,n}^{*-}} - r_{2,n}(t)\Big|_{\tau_{i,n}^{*-}}^{\tau_{i+1,n}^{*+}} + \frac{1}{2} c_v v_{i,n}^{*2} \Big) + V_2^*(t_{n+1}, x_{n+1}).$$

Taking $u^*(t)$ as given, the equilibrium control of Player 2 maximizes the value-to-go for Player 2 for all x, so that the following relations hold:

$$\dot{q}_{2,n}(t) = -w_2 - (a - \frac{b^2}{c_u}\alpha_{1,n}(t))q_{2,n}(t), \forall t \notin \mathcal{T}^n,$$
$$\dot{r}_{2,n}(t) = q_{2,n}(t)\frac{b^2}{c_u}\beta_{1,n}(t), \; \forall t \notin \mathcal{T}^n,$$
$$q_{2,n}(\tau_{i+1,n}^{*-}) = q_{2,n}(\tau_{i+1,n}^{*+}), \; \forall i \in \mathcal{I}^n,$$
$$r_{2,n}(\tau_{i+1,n}^{*-}) = r_{2,n}(\tau_{i+1,n}^{*+}) + \frac{1}{2}\frac{g^2}{c_v}\lambda_2(\tau_{i,n}^{*+})^2 - q_{2,n}(\tau_{i+1,n}^{*+})\lambda_2(\tau_{i,n}^{*+})\frac{g^2}{c_v}, \; \forall i \in \mathcal{I}^n,$$
$$q_{2,n}(t_{n+1}) = q_{2,n+1}(t_{n+1}), \; r_{2,n}(t_{n+1}) = r_{2,n+1}(t_{n+1}), \; q_{2,N}(T) = s_2, \; r_{2,N}(T) = 0,$$

and the profit-to-go is given by

$$V_2^*(t_n, x_n) = q_{2,n}(t_n)x_n + r_{2,n}(t_n),\ \forall n \in \mathcal{N}'. \tag{3.56}$$

Since co-state is equal to the gradient of value function at the sampling instants, we have

$$\lambda_2(t_{n+1}) = q_{2,n+1}(t_{n+1}),\ \forall n \in \mathcal{N}'. \tag{3.57}$$

Using (3.47) in (3.44), we obtain

$$\dot{x}^*(t) = \left(a - \frac{b^2}{c_u}\alpha_{1,n}(t)\right)x^*(t) - \frac{b^2}{c_u}\beta_{1,n}(t) \tag{3.58}$$

$$\Rightarrow x^*(\tau_1^{*-}) = \phi(\tau_{1.n}^{*-}, t_n)x_n + \varphi(\tau_{1,n}^{*-}, t_n), \tag{3.59}$$

where, for $n \in \mathcal{N}'$,

$$\dot{\phi}(t, t_n) = \left(a - \frac{b^2}{c_u}\alpha_{1,n}(t)\right)\phi(t, t_n), \forall t \in (t_n, \tau_{1,n}^*),\ \phi(t_n, t_n) = 1,$$

$$\varphi(\tau_{1,n}^{*-}, t_n) = -\int_{t_n}^{\tau_{1,n}^{*-}} \phi(h, t_n)\frac{b^2}{c_u}\beta_{1,n}(h)dh,$$

$$\dot{\phi}(t, \tau_{i,n}^*) = \left(a - \frac{b^2}{c_u}\alpha_{1,n}(t)\right)\phi(t, \tau_{i,n}^*), \forall t \in (\tau_{i,n}^*, \tau_{i+1,n}^*),\ \phi(\tau_{i,n}^*, \tau_{i,n}^*) = 1, \forall i \in \mathcal{I}^n,$$

$$\varphi(t, \tau_{i,n}^*) = -\int_{\tau_{i,n}^*}^{t} \phi(h, \tau_{i,n}^*)\frac{b^2}{c_u}\beta_{1,n}(h)dh, \forall t \in (\tau_{i,n}^*, \tau_{i+1,n}^*), \forall i \in \mathcal{I}^n,$$

and $\tau_{k_n^*+1,n} := t_{n+1}$. Define

$$x^*(\tau_{i,n}^{*-}) = \phi(\tau_{i,n}^{*-}, t_n)x_n + \varphi(\tau_{i,n}^{*-}, t_n),\ \forall i = \mathcal{I}^n, \tag{3.60a}$$

$$x^*(\tau_{i+1,n}^{*-}) = \phi(\tau_{i+1,n}^{*-}, t_n)x_n + \varphi(\tau_{i+1,n}^{*-}, t_n),\ \forall i = \mathcal{I}^n \backslash \{k_n\}. \tag{3.60b}$$

From (3.58), we obtain

$$\begin{aligned}
x^*(\tau_{i+1,n}^{*-}) =& \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})x^*(\tau_{i,n}^{*+}) + \varphi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+}) \\
&= \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})(x^*(\tau_{i,n}^{*-}) + gv_{i,n}^*)) + \varphi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+}) \\
&= \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\phi(\tau_{i,n}^{*-}, t_n)x_n + \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\varphi(\tau_{i,n}^{*-}, t_n) \\
&\quad + \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})gv_{i,n}^* + \varphi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+}).
\end{aligned}$$

On comparing with (3.60b), we obtain

$$\phi(\tau_{i+1,n}^{*-}, t_n) = \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\phi(\tau_{i,n}^{*-}, t_n),$$

$$\varphi(\tau_{i+1,n}^{*-}, t_n) = \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})\varphi(\tau_{i,n}^{*-}, t_n) + \phi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+})gv_{i,n}^* + \varphi(\tau_{i+1,n}^{*-}, \tau_{i,n}^{*+}).$$

The equilibrium state evolves according to the following equation:

$$\begin{aligned} x(t) =& \phi(t, \tau_{i,n}^*)(\phi(\tau_{i,n}^{*-}, t_n)x_n + gv_{i,n}^* \mathbb{1}_{t>\tau_{i,n}} + \varphi(\tau_{i,n}^{*-}, t_n)) \\ &+ \varphi(t, \tau_{i,n}^*),\ \forall t \in (\tau_{i,n}^*, \tau_{i+1,n}^*), i \in \mathcal{I}^n \cup \{0\}, n \in \mathcal{N}', \end{aligned} \tag{3.61}$$

where $\tau_{0,n}^* := 0$. Then, from (3.43), the equilibrium control of Player 1 is given by

$$\begin{aligned} u^*(t) =& -\frac{b}{c_u}\alpha_{1,n}(t)x(t) + \beta_{1,n}(t) \\ =& -\frac{b}{c_u}\Big(\alpha_{1,n}(t)\big(\phi(t, \tau_{i,n}^{*+})(\phi(\tau_{i,n}^{*-}, t_n)x_n + gv_{i,n}^* \mathbb{1}_{t>\tau_{i,n}} + \varphi(\tau_{i,n}^{*-}, t_n)) \\ &+ \varphi(t, \tau_{i,n}^{*+})\big) + \beta_{1,n}(t)\Big),\ \forall t \in (\tau_{i,n}^*, \tau_{i+1,n}^*), i \in \mathcal{I}^n \cup \{0\}, n \in \mathcal{N}'. \end{aligned} \tag{3.62}$$

References

Aïd, R., Basei, M., Callegaro, G., Campi, L., and Vargiolu, T. (2020). Nonzero-sum stochastic differential games with impulse controls: A verification theorem with applications. *Mathematics of Operations Research*, 45(1):205–232.

Başar, T. (1980). On the existence and uniqueness of closed-loop sampled-data Nash controls in linear-quadratic stochastic differential games. In Iracki, K., Malanowski, K., and Walukiewicz, S., editors, *Optimization Techniques*, pages 193–203, Berlin, Heidelberg. Springer Berlin Heidelberg.

Başar, T. (1991). Optimum H∞ designs under sampled state measurements. *Systems & Control Letters*, 16(6):399 – 409.

Başar, T. (1995). Minimax control of switching systems under sampling. *Systems & Control Letters*, 25(5):315 – 325.

Başar, T. and Olsder, G. J. (1999). *Dynamic Noncooperative Game Theory, 2nd edition*. Society for Industrial and Applied Mathematics, Philadelphia, PA.

Barclay, A., Gill, P. E., and Ben Rosen, J. (1998). Sqp methods and their application to numerical optimal control. In Schmidt, W. H., Heier, K., Bittner, L., and

Bulirsch, R., editors, *Variational Calculus, Optimal Control and Applications*, pages 207–222, Basel. Birkhäuser Basel.

Başar, T. (1989). Time consistency and robustness of equilibria in noncooperative dynamic games. In der Ploeg, F. V. and de Zeeuw, A., editors, *Dynamic Policy Games in Economics*, pages 9 – 54. Springer International Publishing, North-Holland.

Başar, T., Haurie, A., and Zaccour, G. (2018). Nonzero-Sum Differential Games. In Başar, T. and Zaccour, G., editors, *Handbook of Dynamic Game Theory*, pages 61–110. Springer International Publishing, Cham.

Blaquière, A. (1977a). Differential games with piece-wise continuous trajectories. In Hagedorn, P., Knobloch, H. W., and Olsder, G. J., editors, *Differential Games and Applications*, pages 34–69. Springer, Berlin.

Blaquière, A. (1977b). Necessary and sufficient conditions for optimal strategies in impulsive control and application. In Aoki, M. and Morzzolla, A., editors, *New Trends in Dynamic System Theory and Economics*, pages 183–213. Academic Press, New York.

Byrd, R. H., Hribar, M. E., and Nocedal, J. (1999). An interior point algorithm for large-scale nonlinear programming. *SIAM Journal on Optimization*, 9(4):877 – 900.

Büskens, C. and Maurer, H. (2000). Sqp-methods for solving optimal control problems with control and state constraints: adjoint variables, sensitivity analysis and real-time control. *Journal of Computational and Applied Mathematics*, 120(1):85 – 108.

Chahim, M., Hartl, R. F., and Kort, P. M. (2012). A tutorial on the deterministic Impulse Control Maximum Principle: Necessary and sufficient optimality conditions. *European Journal of Operational Research*, 219(1):18–26.

Dockner, E. J., Jorgensen, S., Long, N. V., and Sorger, G. (2000). *Differential Games in Economics and Management Science*. Cambridge University Press.

Drăgan, V., Ivanov, I. G., and Popa, I.-L. (2019). Stochastic linear quadratic differential games in a state feedback setting with sampled measurements. *Systems & Control Letters*, 134:104563.

El Farouq, N., Barles, G., and Bernhard, P. (2010). Deterministic minimax impulse control. *Applied Mathematics and Optimization*, 61(3):353 – 378.

Engwerda, J. C. (2005). *LQ Dynamic Optimization and Differential Games*. Wiley, New York.

Ferrari, G. and Koch, T. (2019). On a strategic model of pollution control. *Annals of Operations Research*, 275(2):297–319.

Geering, H. P. (1976). Continuous-Time Optimal Control Theory for Cost Functionals Including Discrete State Penalty Terms. *IEEE Transactions on Automatic Control*, 21(6):866–869.

Haurie, A., Krawczyk, J. B., and Zaccour, G. (2012). *Games and Dynamic Games*. World Scientific, Singapore.

Sadana, U., Reddy, P. V., and Zaccour, G. (2021). Nash equilibria in nonzero-sum differential games with impulse control. Accepted by European Journal of Operational Research.

Sadana, U., Reddy, P. V., and Zaccour, G. (2020). Open-loop and feedback Nash equilibria in scalar linear-state differential games with impulse control. Les Cahiers du GERAD G-2020-19 GERAD, HEC Montreal, Canada.

Simaan, M. and Cruz Jr., J. (1973). Sampled-data Nash controls in non-zero-sum differential games. *International Journal of Control*, 17(6):1201 – 1209.

Chapter 4

Feedback Nash equilibria in differential games with impulse control

Abstract

We study a class of deterministic finite-horizon two-player nonzero-sum differential games where both players are endowed with a different kind of control. We assume that Player 1 uses piecewise-continuous controls and Player 2 uses impulse controls. For this class of games we seek to derive conditions to characterize the feedback Nash equilibrium strategies of the players. In particular, we show that the number of interventions done by Player 2 have an upper bound. We provide a verification theorem for characterizing the feedback Nash equilibrium strategies using the Hamilton-Jacobi-Bellman (HJB) equations and the quasi-variational inequalities (QVIs). Furthermore, we specialize the obtained results to a linear-quadratic differential game and provide a semi-analytic method for computing the feedback Nash equilibrium.

4.1 Introduction

Many real-world applications such as regulation and cyber-security can be modeled as a two-player finite-horizon nonzero-sum differential game, where one player influences the evolution of the state variable continuously with time whereas the other player introduces jumps in the state variable at certain strategic instants of time. An example of such a setting is a game between an environmental regulation agency that occasionally changes the cap on pollution emissions and a (representative) firm continuously making production decisions with emissions being a by-product.

Our work is closely related to Bertola et al. (2016), Runggaldier and Yasuda (2018) and Aïd et al. (2020). In Bertola et al. (2016) and Runggaldier and Yasuda (2018), the authors study a finite-horizon impulse optimal control problem of a central bank that intervenes in the foreign exchange market and continuously controls its domestic interest rate to keep the exchange rate close to a target value. An extension of this work to a two-player game is given in Aïd et al. (2020) where both players only use impulse controls to keep the state close to their own target values.

Our contribution is four-fold: First, we show under a few regularity assumptions that the number of impulses is bounded by a value that is derived from the problem data. Second, we provide a verification theorem for a general class of differential games with impulse controls that can be used to characterize the feedback Nash equilibrium (FNE). In particular, we show that the (value) functions that satisfy the Hamilton-Jacobi-Bellman equations for Player 1 coupled with a system of quasi-variational inequalities (QVIs) for Player 2 coincide with the respective payoffs of the players in the FNE.

Our third contribution lies in providing conditions for characterizing the FNE in a linear-quadratic differential game (LQDG). LQDGs have been widely studied in engineering, economics and management because they provide a tractable framework to model real-world problems involving non-constant returns to scale, interactions between the players' control variables as well as interactions between the state and control variables. LQDGs assume linear state dynamics, which could be seen as a locally reasonable approximation of non-linear state dynamics. A comprehensive coverage of LQDGs can be found in, e.g., Başar and Olsder (1999), Dockner et al. (2000), Engwerda (2005), Haurie et al. (2012), and Başar et al. (2018). However, these references provide existence and uniqueness results for classical differential games where players only use ordinary controls, and there are no fixed costs in the game. To the best of our knowledge, the literature on differential games does not provide any theoretical or computational means to identify the FNE in nonzero-sum LQDGs with impulse controls.

The specialized linear-quadratic game that we study in this paper involves Player 1 using piecewise-continuous controls to minimize the cost associated with the state deviating from her target value while Player 2 uses impulse controls to instantaneously change the state from one level to another so as to keep the state close to her own target. This model is a multi-agent adaptation of the impulse optimal control problem (single player) studied in Bertola et al. (2016). In particular, in our setting, Player 2's impulse control problem is a modified version of the impulse control problem analyzed in Bertola et al. (2016). Our regularity assumptions on the value function and impulse controls of Player 2 also follow from Bertola et al. (2016) (see also Runggaldier and Yasuda, 2018).

The remainder of the paper is organized as follows. In Section 4.1.1, we give a review of the literature on differential games where at least one player uses piecewise-continuous controls and on impulse games where all players use impulse controls only. We introduce our model in Section 4.2. Further, in Section 4.3, we provide a verification theorem for the existence of FNE. In Section 4.4, we specialize our results to a linear-quadratic game and solve this game in Section 4.5 for two cases. Finally, concluding remarks are given in Section 4.6.

4.1.1 Literature Review

The characterization of optimal impulse control in one decision-maker setting has been the topic of a long series of contributions in diverse domains, e.g, finance (Jeanblanc-Picqué, 1993; Korn, 1998; Cadenillas and Zapatero, 1999; Bertola et al., 2016; Runggaldier and Yasuda, 2018), management (Reddy et al., 2016; Chahim, 2013; Chahim et al., 2017; Erdlenbruch et al., 2013; Sulem, 1986; Bensoussan et al., 2005), epidemiology (Taynitskiy et al., 2019), and medicine (Leander et al., 2015; Hou and Wong, 2011). In contrast, the literature in differential games with impulse controls has been very limited, and predominantly dealt with zero-sum games (see Yong, 1994; Chikrii et al., 2007; Zhang, 2011; Azimzadeh, 2019). With the exception of Sadana et al. (2021, 2020a,b), the equilibrium solutions in nonzero-sum differential games with impulse controls have been obtained under the assumption that the impulse timing is known a priori (see Chang et al., 2013; Zhang, 2011).

The Nash equilibrium varies with the adopted information structure in the game, see Başar and Olsder (1999). In the open-loop information structure, the players' strategies depend only on time and the initial state (which is a known parameter). In Sadana et al. (2021), the authors characterized the open-loop Nash equilibrium (OLNE) for a fairly general class of nonzero-sum differential games with impulse controls and provided an algorithm for computing the equilibrium in LQDGs. Sadana et al. (2020a) characterized the Nash equilibrium in differential games with impulse controls under the sampled-data information structure. Further, Sadana et al. (2020b) determined the FNE for a specialized case of linear-state differential games (LSDGs) with impulse controls, and showed, contrary to the case with ordinary controls, that FNE and OLNE do not coincide. By definition, LSDGs do not account for non-linearities in the state variables or interactions between the state and control variables in the players' objective functionals, which limits their applications in practice. In this paper, we relax this restriction and consider a general class of differential games, and by the same token push further the literature in nonzero-sum differential games.

Finally, we note that there is a class of impulse stochastic games where both players only use impulse controls (see Cosso, 2013; El Asri and Mazid, 2018; Aïd et al., 2020; Ferrari and Koch, 2019). In Aïd et al. (2020), the authors studied infinite-horizon nonzero-sum game problem under the feedback information structure and showed that a system of QVIs gives sufficient conditions for FNE if the value functions of both players satisfy certain regularity conditions. There are no piecewise-continuous controls in their model, which limit their applicability to many problems of interest in regulation and security. Basei et al. (2019) extended their two-player model to a N-player setting and analyzed the corresponding mean-field game. In Campi and De Santis (2020), a game problem between an impulse player and a stopper is solved using the QVIs. The consideration of impulse controls makes it difficult to analytically characterize Nash equilibria for a general class of differential games, which explains why it is tempting to focus on tractable games. For instance, Aïd et al. (2020) determined closed-form solutions for symmetric linear-state impulse stochastic games.

4.2 Model

We consider a deterministic finite-horizon two-player nonzero-sum differential game where the evolution of the state vector is influenced by two different types of control actions of the players. More specifically, the state vector evolves according to the following differential equation due to the actions of Player 1 during the non-impulse instants:

$$\dot{x}(t) = f(x(t), u(t)),\ x(0^-) = x_0,\ \text{for } t \neq \{\tau_1, \tau_2, ..., \tau_k\}, \tag{4.1}$$

where $u(t) \in \Omega_u \subset \mathbb{R}^{m_1}$, $f : \mathbb{R}^n \times \mathbb{R}^{m_1} \to \mathbb{R}^n$. The action profiles of Player 1 for all $t \in [0, T]$ are given by $u(.)$. At the impulse instants $\{\tau_1, \tau_2, \cdots, \tau_k\}$, Player 2 gives impulses $v_i \in \Omega_v \subset \mathbb{R}^{m_2}$ to cause jumps in the state

$$x(\tau_i^+) - x(\tau_i^-) = g(x(\tau_i^-), v_i),\ \text{for } i = \{1, 2, ..., k\}, \tag{4.2}$$

where $x(\tau_i^-) = \lim_{t\uparrow\tau_i} x(t)$, $x(\tau_i^+) = \lim_{t\downarrow\tau_i} x(t)$ and $g : \mathbb{R}^n \times \mathbb{R}^{m_2} \to \mathbb{R}^n$. The control actions of Player 2 during the game are denoted by $\tilde{v} = ((\tau_1, v_1), (\tau_2, v_2), \ldots, (\tau_k, v_k), k)$ where $k \in \mathbb{N}$ (the set of natural numbers). The number of impulses k is also a decision variable of Player 2. The control sets Ω_u and Ω_v are assumed to be bounded and convex open sets.

Player 1 and Player 2 minimize their respective objective functions that are given by

$$J_1(x_0, u(.), \tilde{v}) = \int_0^T h_1(x(t), u(t))dt + \sum_{i=1}^{k} \mathbb{1}_{0\leq\tau_i<T}\, b_1(x(\tau_i^-), v_i) + s_1(x(T)), \tag{4.3}$$

$$J_2(x_0, u(.), \tilde{v}) = \int_0^T h_2(x(t), u(t))dt + \sum_{i=1}^{k} \mathbb{1}_{0\leq\tau_i<T}\, b_2(x(\tau_i^-), v_i) + s_2(x(T)), \tag{4.4}$$

where $h_i : \mathbb{R}^n \times \mathbb{R}^{m_1} \to \mathbb{R}$ is the running cost of Player i, $b_i : \mathbb{R}^n \times \mathbb{R}^{m_2} \to \mathbb{R}$ is the cost accrued by Player i at the time of impulse, and $s_i : \mathbb{R}^n \to \mathbb{R}$ is the terminal cost of Player i. Here, $\mathbb{1}_y$ denotes an indicator function of y, that is, $\mathbb{1}_y$ is equal to 1 if y holds; otherwise, it is equal to 0.

We make the following assumptions regarding the sate dynamics (4.1)–(4.2) and the objectives in (4.3)–(4.4):

Assumption 4.1 *(i) $f(x,u)$ is bounded and Lipschitz continuous such that, for $c_f > 0$, we have*

$$|f(x,u)| \leq c_f,\ \forall (x,u) \in \mathbb{R}^n \times \Omega_u,$$
$$|f(x,u) - f(y,u)| \leq c_f|x-y|,\ \forall x,y \in \mathbb{R}^n,\ u \in \Omega_u.$$

(ii) $g(x,v)$ is bounded and Lipschitz continuous such that, for $c_g > 0$, we have

$$|g(x,v)| \leq c_g,\ \forall (x,v) \in \mathbb{R}^n \times \Omega_v,$$
$$|g(x,v) - g(y,v)| \leq c_g|x-y|\ \forall x,y \in \mathbb{R}^n,\ v \in \Omega_v.$$

(iii) For $i = \{1,2\}$, $h_i(x,u)$ and $b_i(x,v)$ are bounded such that, for $c_{h_i} > 0$ and $c_{b_i} > 0$, we have

$$|h_i(x,u)| \leq c_{h_i},\ \forall (x,u) \in \mathbb{R}^n \times \Omega_u,$$
$$|b_i(x,v)| \leq c_{b_i},\ \forall (x,v) \in \mathbb{R}^n \times \Omega_v.$$

(iv) $\forall x \in \mathbb{R}^n$, $\inf_{v \in \Omega_v} b_2(x,v) = \mu > 0$.

(v) The salvage values $s_i(x)$ are bounded, such that, for $c_{s_i} > 0$, $|s_i(x)| \leq c_{s_i}$, $\forall x \in \mathbb{R}^n$.

Assumption 1.(i) and 1.(ii) ensure that there exists a unique state trajectory $x(\cdot)$ for any measurable $u(\cdot)$ and impulse sequence $\{(\tau_i, v_i)\}_{i=1}^{k}$. Assumption 1.(iv) ensures that Player 2 intervenes only a finite number of times in the game due to the fixed cost associated with each impulse (see Bertola et al., 2016). Assumptions (iii) and (v) are used later to show that the value functions of Player 1 and Player 2 are bounded.

4.2.1 Feedback Nash equilibrium

We focus our attention on the derivation of Nash equilibrium under memoryless perfect state information structure. The strategy spaces of the players under the memoryless perfect state information structure are defined as follows: Let $\Sigma := \{(t,x) \,|\, t \in [0,T], x \in \mathbb{R}^n\}$ and let $\mathcal{T}$ denote the set of admissible impulse instants. A feedback (or Markovian) strategy selects the control action according to a feedback rule, i.e., a mapping from the state space into the action set. In our setting, this implies that Player 1's controls at time $t \in [0,T]\backslash\mathcal{T}$ are given by $u(t) := \gamma(t, x(t)) \in \Omega_u$, where $\gamma : [0,T]\backslash\mathcal{T} \times \mathbb{R}^n \to \Omega_u$ is a measurable mapping, and the set of all such mappings is denoted by Γ. Similarly, a strategy of Player 2 is given by $\delta = (\mathcal{C}, v)$ where $\mathcal{C}$ is a fixed open subset of Σ and v is a continuous function from Σ to Ω_v. The strategies of the players have the following interpretation: Player 1 continuously controls the state trajectory using state feedback $\gamma(t,x)$ during the time the state lies in $\mathcal{C}$. Once the state leaves set $\mathcal{C}$, Player 2 intervenes and gives an impulse of size v such that the next state $x+v$ lies in set $\mathcal{C}$.

Definition 4.1 *The sequence $\tilde{v} = \{(\tau_1, v_1), (\tau_2, v_2), \ldots, (\tau_k, v_k), k\}$, is an admissible impulse control sequence of Player 2 if the number of impulses is finite and the impulse instants lie in the set $\mathcal{T}$ given by*

$$\mathcal{T} = \{\tau_i,\ i = 1, 2, \cdots, k \mid 0 \le \tau_1 < \tau_2 \cdots < \tau_k < T, k < \infty\},$$
$$\tau_n = \inf\{t > \tau_{n-1} : (t,x) \notin \mathcal{C}\}, \tau_0 := 0.$$

Using the strategies of the players, we can rewrite the objective functions of Player 1 and Player 2 at any $(t,x) \in \Sigma$ as follows:

$$J_1(x, \gamma_{[t,T]}, \delta_{[t,T]}) = \int_t^T h_1(x(s), \gamma(s, x(s)))ds + \sum_{j=i}^{k} \mathbb{1}_{t \le \tau_j < T}\ b_1(x(\tau_j^-), v_j) \\ + s_1(x(T)), \tag{4.5}$$

$$J_2(x, \gamma_{[t,T]}, \delta_{[t,T]}) = \int_t^T h_2(x(s), \gamma(s, x(s)))ds + \sum_{j=i}^{k} \mathbb{1}_{t \le \tau_j < T}\ b_2(x(\tau_j^-), v_j) \\ + s_2(x(T)), \tag{4.6}$$

where $\gamma_{[t,T]} \in \Gamma_{[t,T]}$ and $\delta_{[t,T]} \in \Delta_{[t,T]}$ are restrictions of γ and δ, respectively, to the interval $[t,T]$, and $\Gamma_{[t,T]}$ and $\Delta_{[t,T]}$ denote the strategy sets for Player 1 and Player 2, respectively, in the interval $[t,T]$. The state dynamics are given by

$$\dot{x}(t) = f(x(t), \gamma(t, x(t))), t \neq \{\tau_i, \tau_{i+1}, ..., \tau_k\}, x(t) = x, \tag{4.7}$$
$$x(\tau_j^+) - x(\tau_j^-) = g(x(\tau_j^-), v_j),\ j = \{i, i+1, \cdots, k\}. \tag{4.8}$$

The feedback Nash equilibrium is defined as follows:

Definition 4.2 *For the differential game described by (4.5–4.8) with memoryless perfect state information pattern, the strategy profile* $(\gamma^*, \delta^*) \in \Gamma \times \Delta$ *constitutes a feedback Nash equilibrium solution if there exists value functionals* $V_l(.,.)$ *defined on* $[0,T] \times \mathbb{R}^n$ *and satisfying the following relations for each player* $l \in \{1,2\}$*:*

$$V_1(T,x) = s_1(x), \tag{4.9}$$

$$\begin{aligned} &V_1(t,x) \\ &= \int_t^T h_1(x^*(s), \gamma^*(s, x^*(s)))ds + \sum_{j=i}^{k} \mathbb{1}_{t \le \tau_j^* < T}\, b_1(x^*(\tau_j^{*-}), v_j^*) + s_1(x^*(T)) \\ &\le \int_t^T h_1(x_1(s), \gamma(s, x_1(s)))ds + \sum_{j=i}^{k} \mathbb{1}_{t \le \tau_j^* < T}\, b_1(x_1(\tau_j^{*-}), v_j^*) + s_1(x_1(T)) \qquad (4.10) \\ &\qquad \forall \gamma \in \Gamma_{[t,T]},\ x \in \mathbb{R}^n, \end{aligned}$$

$$V_2(T,x) = s_2(x), \tag{4.11}$$

$$\begin{aligned} &V_2(t,x) \\ &= \int_t^T h_2(x^*(s), \gamma^*(s, x^*(s)))ds + \sum_{j=i}^{k} \mathbb{1}_{t \le \tau_j^* < T}\, b_2(x^*(\tau_j^{*-}), v_j^*) + s_2(x^*(T)) \\ &\le \int_t^T h_2(x_2(s), \gamma^*(s, x_2(s)))ds + \sum_{j=i}^{k} \mathbb{1}_{t \le \tau_j < T}\, b_2(x_2(\tau_j^{-}), v_j) + s_2(x_2(T)) \qquad (4.12) \\ &\qquad \forall \delta \in \Delta_{[t,T]},\ x \in \mathbb{R}^n, \end{aligned}$$

where on the interval $[t,T]$*,*

$$\begin{aligned} \dot{x}_1(s) &= f(x_1(s), \gamma(s, x_1(s))),\ x_1(t) = x, \textit{ for } s \neq \{\tau_i^*, \tau_{i+1}^*, \cdots, \tau_k^*\}, \\ x_1(\tau_j^{*+}) &= x_1(\tau_j^{*-}) + g(x_1(\tau_j^{*-}), v_j^*), \textit{ for } j = \{i, i+1, \cdots, k\}, \end{aligned}$$

$$\begin{aligned} \dot{x}_2(s) &= f(x_2(s), \gamma^*(s, x_2(s))),\ x_2(t) = x, \textit{ for } s \neq \{\tau_i, \tau_{i+1}, \cdots, \tau_k\}, \\ x_2(\tau_j^{+}) &= x_2(\tau_j^{-}) + g(x_2(\tau_j^{-}), v_j), \textit{ for } j = \{i, i+1, \cdots, k\}, \end{aligned}$$

$$\begin{aligned} \dot{x}^*(s) &= f(x^*(s), \gamma^*(s, x^*(s))),\ x(t) = x, \textit{ for } s \neq \{\tau_i^*, \tau_{i+1}^*, \cdots, \tau_k^*\}, \\ x^*(\tau_j^{*+}) &= x(\tau_j^{*-}) + g(x(\tau_j^{*-}), v_j^*), \textit{ for } j = \{i, i+1, \cdots, k\}. \end{aligned}$$

Feedback Nash equilibria satisfy a useful property referred to as strong time-consistency, which is described as follows. Let $D(\Xi, [0,T])$ denote the two-player differential game with impulse controls where Ξ is the product strategy space and

$[0\ T]$ is the time horizon of the game such that

$$(\gamma,\delta)_{[s,t]} \in \Xi_{[s,t]}, \text{and } \gamma_{[s,t]} \in \Gamma_{[s,t]} \text{ and } \delta_{[s,t]} \in \Delta_{[s,t]}$$

denote the truncations of $\gamma \in \Gamma$ and $\delta \in \Delta$ to the interval $[s,t] \subset [0,T]$, and $(\gamma,\delta)_{[s,t]}$ is a shorthand notation for $(\gamma_{[s,t]}, \delta_{[s,t]})$. We denote the truncated game corresponding to $D(\Xi,[0,T])$ as follows:

$$\begin{aligned} D^{[s,t]}_{(\alpha^1,\alpha^2)} = D(\big\{(\gamma,\delta) \in \Xi | (\gamma,\delta)_{[0,s)} = (\alpha^1,\alpha^2)_{[0,s)}, (\gamma,\delta)_{(t,T]} = (\alpha^1,\alpha^2)_{(t,T]}, \\ (\gamma,\delta)_{[s,t]} \in (\Gamma,\Delta)_{[s,t]}\big\}; [0,T]), \end{aligned}$$

where the players' policies are fixed in the interval $[0,s)$ and $(t,T]$ as $\alpha^i_{[0,s)}$, $\alpha^i_{[t,T]}$ for $i = \{1,2\}$.

Definition 4.3 (Strongly time-consistent equilibrium) *A pair of policies $(\gamma^*,\delta^*) \in (\Gamma,\Delta)$ that solve the differential game $D(\Xi,[0,T])$ is strongly time consistent if its truncation to the interval $[s,T]$, $(\gamma^*_{[s,T]}, \delta^*_{[s,T]})$, solves the subgame $D^{[s,T]}_{(\alpha^1,\alpha^2)}$ for every $(\alpha^1,\alpha^2)_{[0,s)} \in \Xi_{[0,s)}$ and for all $s \in [0,T]$ (see Başar and Olsder, 1999).*

4.3 Verification theorem

The differential game (4.5)-(4.8) comprises of a non-standard optimal control problem of Player 1 due to intervention costs and state jumps, and an impulse optimal control problem of Player 2. To characterize the feedback Nash equilibrium strategies, we make the following assumption:

Assumption 4.2 *There exists a unique, finite, measurable function $v : [0,T] \times \mathbb{R}^n \to \Omega_v$ (see Aïd et al., 2020; Bertola et al., 2016) such that*

$$v(t,x) = \arg\min_{\eta \in \Omega_v} \{V_2(x + g(x,\eta)) + b_2(x,\eta)\}. \tag{4.13}$$

(4.13) gives the optimal impulse level at any (t,x) since it minimizes the sum of immediate cost ($b_2(x,\eta)$) incurred by giving an impulse of size η and the cost-to-go by playing optimally afterwards.

Let V_1 be the Nash equilibrium payoff of Player 1. Suppose the equilibrium strategy of Player 2 is $\delta^* = (\mathcal{C}, v)$ for which the equilibrium timing and level of impulses are given by the sequence $\{(\tau_i^*, v_i^*), i = 1, 2, \cdots, k\}$. For Player 1, the sufficient conditions for the existence of Nash equilibrium are given by

$$-\frac{\partial V_1(t,x)}{\partial t} = \min_{u \in \Omega_u} \left(h_1(x(t), u(t)) + \left(\frac{\partial V_1}{\partial x} \right)^T f(x(t), u(t)) \right), \ (t,x) \in \mathcal{C}, \quad \text{(4.14a)}$$

$$V_1(T, x(T)) = s_1(x(T)), \ \forall (T, x) \in \Sigma, \quad \text{(4.14b)}$$

$$V_1(\tau_i^{*-}, x(\tau_i^{*-})) = V_1(\tau_i^{*+}, x(\tau_i^{*+})) + b_1(x(\tau_i^{*-}), v_i^*), \quad \text{(4.14c)}$$
$$(\tau_i^{*-}, x(\tau_i^{*-})) \in \Sigma \backslash \mathcal{C}.$$

A formal proof of sufficiency of the above conditions is given in Theorem 4.1 which can be interpreted as follows. An admissible impulse cannot occur at the terminal time hence condition (4.14b) holds. In the continuation region $\mathcal{C}$, Player 2 does not give any impulse and therefore, the value function of Player 1 satisfies the Hamilton-Jacobi-Bellman equation (4.14a). When an impulse occurs in the intervention region, that is, $(\tau_i^{*-}, x(\tau_i^{*-}) \in \Sigma \backslash \mathcal{C}$, then Player 1's equilibrium cost-to-go is the sum of the additional cost, $b_1(x(\tau_i^{*-}), v_i^*)$, incurred due to the intervention by Player 2 and the equilibrium cost-to-go by playing optimally afterwards.

The optimal cost-to-go by giving an optimal impulse of size v^* in (4.13) at (t, x) can be written using the intervention operator $\mathcal{R}$ as follows:

$$\mathcal{R}V_2(t,x) = V_2(x + g(x, v^*)) + b_2(x, v^*). \quad \text{(4.15)}$$

For a given equilibrium strategy γ^* of Player 1, the value function $V_2(t, x) : \Sigma \to \mathbb{R}$ satisfies the (weak) QVIs if

$$\frac{\partial V_2(t,x)}{\partial t} + \mathcal{H}_2(x, \gamma^*(t,x), \frac{\partial V_2(t,x)}{\partial x}) \geq 0, \ \forall t \in [0,T], \text{ a.a. } x \in \mathbb{R}^n, \quad \text{(4.16a)}$$

$\forall (t, x) \in \Sigma,$ the following two relations hold,

$$V_2(t,x) \leq \mathcal{R}V_2(t,x), \quad \text{(4.16b)}$$

$$(V_2(t,x) - \mathcal{R}V_2(t,x)) \left(\frac{\partial V_2(t,x)}{\partial t} + \mathcal{H}_2(x, \gamma^*(t,x), \frac{\partial V_2(t,x)}{\partial x}) \right) = 0, \quad \text{(4.16c)}$$

and $V_2(T, x) = s_2(x(T)), \forall (T, x) \in \Sigma,$ (4.16d)

where

$$\mathcal{H}_2(x, \gamma^*(t,x), \frac{\partial V_2(t,x)}{\partial x}) = h_2(x, \gamma^*(t,x)) + \left(\frac{\partial V_2(t,x)}{dx} \right)^T f(x, \gamma^*(t,x)). \quad \text{(4.16e)}$$

Condition (4.16b) ensures that the value function is at most equal to the optimal cost-to-go by giving an impulse. Clearly, Player 2 does not give an impulse when the value function is strictly less than the cost-to-go by giving an impulse. Hence, when $V_2(t,x) = \mathcal{R}V_2(t,x)$, Player 2 gives an impulse. At any (t,x), condition (4.16c) ensures either player 2 waits so that the HJB like equation (4.16a) for Player 2 holds with equality or Player 2 gives an impulse. This allows us to define the continuation and intervention sets for Player 2 as follows:

Definition 4.4 *The continuation and intervention sets are given by*

$$\mathcal{C} = \left\{(t,x) \in \Sigma | V_2(t,x) < \mathcal{R}V_2(t,x), \frac{\partial V_2(t,x)}{\partial t} + \mathcal{H}_2(x, \gamma^*(t,x), \frac{\partial V_2(t,x)}{\partial x}) = 0\right\}, \tag{4.17}$$

$$\mathcal{I} = \left\{(t,x) \in \Sigma | V_2(t,x) = \mathcal{R}V_2(t,x), \frac{\partial V_2(t,x)}{\partial t} + \mathcal{H}_2(x, \gamma^*(t,x), \frac{\partial V_2(t,x)}{\partial x}) \geq 0\right\}. \tag{4.18}$$

Next, we show that there can only be a finite number of impulses during the game.

Proposition 4.1 *Let Assumption 4.1 hold. Then the value functions of Player 1 and Player 2 are bounded. The equilibrium number of impulses $K \in \mathbb{N}$ is uniformly bounded by*

$$K = \lceil \frac{2\,(Tc_{h_2} + c_{s_2})}{\mu} \rceil, \tag{4.19}$$

where $\mu = \inf_{z\in\Omega_v} b_2(x,z) > 0,\ \forall x \in \mathbb{R}^n$, and $\lceil y \rceil$ denotes the smallest integer that is greater than or equal to y.

Proof. See Appendix 4.7.1. ■

The sufficient conditions to characterize the FNE of the differential game described in (4.5)-(4.8) are given in the next theorem.

Theorem 4.1 (Verification Theorem) *Let $V_i : \Sigma \to \mathbb{R}, i = 1, 2$ be two functions. Let Assumptions 4.1 and 4.2 hold. Suppose V_1 satisfies* (4.14a)-(4.14c) *and V_2 satisfies the QVIs (4.16a-4.16d), and there exists a function $\gamma^*(t,x) = u(t)$ such that $u(t)$ minimizes the expression on the right-hand side of* (4.14a). *Let there exist a function $\delta^* = (\mathcal{C}, v)$ such that equilibrium impulses occur at $\{\tau_1^*, \tau_2^*, \cdots, \tau_k^*\}$ and the corresponding impulse* *levels* $\{v_1^*, v_2^*, \cdots, v_k^*\}$ *minimize the expression on the right-hand side of* (4.13). *Then,* γ^* *and* δ^* *are the feedback Nash equilibrium strategies of Player 1 and Player 2, respectively.*

Proof. Let γ^* be the equilibrium strategy of Player 1. In the continuation region C, we use the Taylor series expansion of $V_2(t, x)$ to obtain

$$\begin{aligned} & V_2(T, x(T)) - V_2(t, x) \\ & = \sum_{j=i}^{k} \int_{t\wedge\tau_j}^{t\wedge\tau_{j+1}} \left(\frac{\partial V_2(s, x_2(s))}{\partial s} + \left(\frac{\partial V_2(s, x_2(s))}{\partial x} \right)^T f(x_2(s), \gamma^*(s, x_2(s))) \right) ds \\ & + \sum_{j=i}^{k} \mathbb{1}_{t\leq\tau_j<T} (V_2(\tau_j, x_2(\tau_j^+)) - V_2(\tau_j, x_2(\tau_j^-))). \end{aligned} \tag{4.20}$$

The value function satisfies (4.16a) for all $(t, x) \in \Sigma$, so we have

$$\begin{aligned} & \frac{\partial V_2}{\partial t}(s, x_2(s)) + \left(\frac{\partial V_2}{\partial x}(s, x_2(s)) \right)^T f(x_2(s), \gamma(s, x_2(s))) \\ & \quad \geq -h_2(x_2(s), \gamma^*(s, x_2(s)). \end{aligned} \tag{4.21}$$

From (4.16b), we obtain

$$\begin{aligned} & V_2(\tau_j, x_2(\tau_j^+)) - V_2(\tau_j, x_2(\tau_j^-)) \\ & = \mathcal{R}V_2(\tau_j, x_2(\tau_j^-)) - V_2(\tau_j, x_2(\tau_j^-)) - b_2(x_2(\tau_j^-), v_j) \geq -b_2(x_2(\tau_j^-), v_j). \end{aligned} \tag{4.22}$$

Substitute (4.21) and (4.22) in (4.20) to obtain

$$V_2(T, x(T)) - V_2(t, x) \geq \sum_{j=i}^{k} \int_{t\wedge\tau_j}^{t\wedge\tau_{j+1}} -h_2(x(s), \gamma(s, x(s))ds - \sum_{j=i}^{k} b_2(x(\tau_j^-), v_j).$$

Substituting $V_2(T, x) = s_2(x)$ in the above inequality yields

$$\begin{aligned} V_2(t, x) \leq & \sum_{j=i}^{k} \int_{t\wedge\tau_j}^{t\wedge\tau_{j+1}} h_2(x_2(s), \gamma^*(s, x_2(s))ds + \sum_{j=i}^{k} b_2(x_2(\tau_j^-), v_j) + s_2(x_2(T)) \\ & = J_2(x, \gamma^*, \delta). \end{aligned}$$

The above relation holds with equality for equilibrium strategy δ^* of Player 2 when the value function V_2 satisfies the QVIs (4.16a-4.16d).

Next, we verify the sufficient conditions for Player 1 taking the equilibrium strategy of Player 2 as δ^*. Using the Taylor series expansion of V_1 between the impulse instants $(\tau_i^*, \tau_{i+1}^*), i = \{1, 2, \cdots, k\}$,

$$V_1(T, x(T)) - V_1(t, x)$$

$$= \sum_{j=i}^{k} \int_{t\wedge\tau_j^*}^{t\wedge\tau_{j+1}^*} \left(\frac{\partial V_1}{\partial t}(s, x_1(s)) + \left(\frac{\partial V_1}{\partial x}(s, x_1(s)) \right)^T f(x_1(s), u(s)) \right) ds$$

$$+ \sum_{j=i}^{k} \mathbb{1}_{t \le \tau_j^* < T} (V_1(\tau_j^{*+}, x_1(\tau_j^{*+})) - V_1(\tau_j^{*-}, x_1(\tau_j^{*-})))$$

$$\ge - \sum_{j=i}^{k} \int_{t\wedge\tau_j^*}^{t\wedge\tau_{j+1}^*} h_1(x_1(s), u(s)) ds$$

$$+ \sum_{j=i}^{k} \mathbb{1}_{t \le \tau_j^* < T} (V_1(\tau_j^{*}, x_1(\tau_j^{*+})) - V_1(\tau_j^{*}, x_1(\tau_j^{*-}))).$$

where the last inequality follows from (4.14a). From the terminal condition on V_1 and additional cost incurred due to the impulse control of Player 2, we obtain

$$V_1(t, x) \le \sum_{j=i}^{k} \int_{t\wedge\tau_j^*}^{t\wedge\tau_{j+1}^*} h_1(x_1(s), u(s)) ds + \sum_{i=1}^{k} \mathbb{1}_{t \le \tau_j^* < T}\, b_1(x_1(\tau_j^-), v_j^*) + s_2(x(T))$$
$$= J_1(x(t), \gamma, \delta^*).$$

The above inequality becomes an equality if the value function of Player 1 satisfies (4.14a)-(4.14c) for the equilibrium strategy γ^*. ■

QVIs have been solved in the literature under some restrictive assumptions on the value functions even for games with linear objective functions (see Aïd et al., 2020; Campi and De Santis, 2020). An additional difficulty in our case is that the QVIs are coupled with the equilibrium conditions for Player 1 who has piecewise-continuous controls. In the next section, we specialize our results to linear-quadratic differential games and provide semi-analytical solutions.

4.4 A linear-quadratic differential game with targets

In this section, we consider a one-dimensional two-player linear-quadratic differential game where Player 1 uses piecewise-continuous controls and Player 2 uses impulse controls. Player 1 and Player 2 aim to minimize the costs resulting from the deviation of the state away from their target state values ρ_1 and ρ_2, respectively. In the following formulation of the linear-quadratic differential game with impulse controls (iLQDG), the structure of Player 2's problem (objective functions and state dynamics) is an adpatation of the impulse optimal control problem analyzed in Bertola et al. (2016).

$$\text{(iLQDG)}\quad J_1(x_0, u(\cdot), \tilde{v}) = \int_0^T \frac{1}{2}\left(w_1(x(t)-\rho_1)^2 + 2w_{11}(x(t)-\rho_1(t)) + r_1 u(t)^2\right) dt$$
$$+ \sum_{i=1}^{k} z_1(x(\tau_i^+) - x(\tau_i^-)) + \frac{1}{2} s_1(x(T)-\rho_1)^2, \tag{4.23a}$$

$$J_2(x_0, u(\cdot), \tilde{v}) = \int_0^T \frac{1}{2} w_2(x(t)-\rho_2)^2 dt + \sum_{i=1}^{k} h(v_i) + \frac{1}{2} s_2(x(T)-\rho_2)^2, \tag{4.23b}$$

$$\dot{x}(t) = ax(t) + bu(t),\ x(0^-) = x_0,\ \forall t \neq \{\tau_1, \tau_2, ..., \tau_k\}, \tag{4.23c}$$

$$x(\tau_i^+) = x(\tau_i^-) + v_i, \forall i = \{1, 2, ..., k\}, \tag{4.23d}$$

where

$$h(v_i) := \begin{cases} C + cv_i & \text{if } v_i > 0 \\ \min(C, D) & \text{if } v_i = 0 \\ D - dv_i & \text{if } v_i < 0, \end{cases} \tag{4.24}$$

and $w_1,\ w_{11},\ r_1,\ z_1,\ s_1,\ w_2,\ s_2,\ C,\ D,\ c,\ d$ are positive constants.
Feedback Nash equilibrium computation involves solving the control problem of each player for a given equilibrium strategy of the other player.

Assumption 4.3 *Player* 2 *gives an impulse if* (t, x) *does not lie in the continuation set* $\mathcal{C}$ *given by*

$$\mathcal{C} = \{(t, x) \in \Sigma \mid \ell_1(t) < x < \ell_2(t)\}. \tag{4.25}$$

Player 2 shifts the state to $\alpha(t)$ *if* $x \leq \ell_1(t)$ *and to* $\beta(t)$ *if* $x \geq \ell_2(t)$ *so that the following relation holds:*

$$\ell_1(t) < \alpha(t) < \beta(t) < \ell_2(t). \tag{4.26}$$

The above assumption has also been made in the impulse optimal control literature when a decision-maker uses threshold-type impulse controls due to the fixed costs associated with the impulse controls (see Bertola et al., 2016; Runggaldier and Yasuda, 2018).

4.4.1 Optimal control problem of Player 1

Let the equilibrium strategy of Player 2 be given by δ^* such that Player 2 gives an impulse if the state leaves the continuation set $\mathcal{C}$. Then the equilibrium strategy of Player 1 can be determined by finding the value function that satisfies (4.14a)-(4.14c) for the iLQDG given in (4.23). Since the game is linear-quadratic, we make the following informed guess on the form of the value function of Player 1 (see Bertola et al., 2016):

$$V_1(t,x) = \frac{1}{2}p_1(t)x^2 + q_1(t)x + n_1(t). \tag{4.27}$$

From (4.14a), we have

$$-\frac{\partial V_1(t,x)}{\partial t} = \min_{u\in\Omega_u}\left(\frac{1}{2}w_1x^2 + \frac{1}{2}r_1u(t)^2 + \left(\frac{\partial V_1}{\partial x}\right)(ax + bu(t))\right). \tag{4.28}$$

Differentiating the right-hand side of the above equation and equating the result to zero yields the equilibrium strategy of Player 1

$$\gamma^*(t,x) = u^*(t) = -\frac{b}{r_1}\left(\frac{\partial V_1}{\partial x}\right) = -\frac{b}{r_1}(p_1(t)x + q_1(t)). \tag{4.29}$$

Substituting (4.29) in the state dynamics (4.23c), we obtain

$$\begin{aligned}\dot{x}(t) &= ax(t) + bu^*(t) = ax(t) - \frac{b^2}{r_1}\left(p_1(t)x(t) + q_1(t)\right)\\ &= \left(a - \frac{b^2}{r_1}p_1(t)\right)x(t) - \frac{b^2}{r_1}q_1(t)\\ &= a_x(t)x(t) + b_xq_1(t),\end{aligned} \tag{4.30}$$

where $a_x(t) = a - \frac{b^2}{r_1}p_1(t)$ and $b_x = -\frac{b^2}{r_1}$. On substituting (4.29) and (4.27) in (4.28), we obtain

$$\begin{aligned}-\frac{1}{2}\dot{p}_1(t)x^2 - \dot{q}_1(t)x - \dot{n}_1(t) &= \frac{1}{2}w_1(x-\rho_1)^2 + w_{11}(x-\rho_1) - \frac{1}{2}b_x(p_1(t)x + q_1(t))^2\\ &\quad + (p_1(t)x + q_1(t))\left(a_x(t)x + b_xq_1(t)\right)\\ \Rightarrow -\dot{p}_1(t)x^2 - 2\dot{q}_1(t)x - 2\dot{n}_1(t) &= w_1x^2 + w_1\rho_1^2 + 2x(w_{11} - w_1\rho_1) - 2w_{11}\rho_1\\ &\quad - b_x\left(p_1(t)^2x^2 - q_1(t)^2\right) + 2a_x(t)\left(p_1(t)x + q_1(t)\right)x.\end{aligned}$$

Upon rearranging a few terms in the above equation, we get

$$\left(\dot{p}_1(t) + w_1 + b_xp_1(t)^2 + 2p_1(t)a\right)x^2 + w_1\rho_1^2 + 2\dot{n}_1(t) + b_xq_1(t)^2 - 2w_{11}\rho_1$$

$$+\left(2\dot{q}_1(t)+2a_x(t)q_1(t)-2w_1\rho_1+2w_{11}\right)x=0.$$

Since the above equation must hold for all x except at $(t,x)\notin\mathcal{C}$, $p_1(\cdot)$, $q_1(\cdot)$, and $n_1(\cdot)$ evolve as follows:

$$\dot{p}_1(t)=-w_1-b_xp_1(t)^2+2p_1(t)a,\ (t,x)\notin\mathcal{C},p_1(T)=s_1, \tag{4.31a}$$

$$\dot{q}_1(t)=-a_x(t)q_1(t)+w_1\rho_1-w_{11},\ (t,x)\notin\mathcal{C},\ q_1(T)=-s_1\rho_1, \tag{4.31b}$$

$$\dot{n}_1(t)=-\frac{1}{2}b_xq_1(t)^2-\frac{w_1\rho_1^2}{2}+w_{11}\rho_1,\ (t,x)\notin\mathcal{C},\ n_1(T)=\frac{1}{2}s_1\rho_1^2. \tag{4.31c}$$

When an impulse occurs, that is, $(\tau_i^*,x(\tau_i^*))\in\Sigma\backslash\mathcal{C}$, the following relation holds for the value function of Player 1:

$$\begin{aligned}&\frac{1}{2}p_1(\tau_i^{*-})x(\tau_i^{*-})^2+q_1(\tau_i^{*-})x(\tau_i^{*-})+n_1(\tau_i^{*-})\\&\quad=\frac{1}{2}p_1(\tau_i^{*+})(x(\tau_i^{*-})+v_i^*)^2+q_1(\tau_i^{*+})(x(\tau_i^{*-})+v_i^*)+n_1(\tau_i^{*+})\\&\qquad+z_1(x(\tau_i^{*+})-x(\tau_i^{*-})).\end{aligned}$$

The equilibrium strategy of Player 2 is to bring the state to $\alpha(t)$ if $x(t)\le\ell_1(t)$ and to $\beta(t)$ if $x(t)\ge\ell_2(t)$, that is, $x(\tau_i^{*-})+v_i^*=\alpha(\tau_i^{*-})$ if $x(t)\le\ell_1(t)$ and $x(\tau_i^{*-})+v_i^*=\beta(\tau_i^{*-})$ if $x(t)\ge\ell_2(t)$. Therefore, we have

$$\begin{aligned}&\frac{1}{2}p_1(\tau_i^{*-})x(\tau_i^{*-})^2+q_1(\tau_i^{*-})x(\tau_i^{*-})+n_1(\tau_i^{*-})\\&\quad=\frac{1}{2}p_1(\tau_i^{*+})\alpha(\tau_i^{*-})^2+(z_1+q_1(\tau_i^{*+}))\alpha(\tau_i^{*-})+n_1(\tau_i^{*+})-z_1x(\tau_i^{*-}),\ x\le\ell_1(t),\\&\frac{1}{2}p_1(\tau_i^{*-})x(\tau_i^{*-})^2+q_1(\tau_i^{*-})x(\tau_i^{*-})+n_1(\tau_i^{*-})\\&\quad=\frac{1}{2}p_1(\tau_i^{*+})\beta(\tau_i^{*-})^2+(z_1+q_1(\tau_i^{*+}))\beta(\tau_i^{*-})+n_1(\tau_i^{*+})-z_1x(\tau_i^{*-}),\ x\ge\ell_2(t).\end{aligned}$$

Since the above two equations hold for all $x\le\ell_1(t)$ and $x\ge\ell_2(t)$, respectively, we have

$$\begin{aligned}p_1(\tau_i^{*-})&=0,\\q_1(\tau_i^{*-})&=-z_1,\\n_1(\tau_i^{*-})&=n_1(\tau_i^{*+})+\frac{1}{2}p_1(\tau_i^{*+})[\mathbb{1}_{x\le\ell_1(\tau_i^{*-})}\alpha(\tau_i^{*-})^2+\mathbb{1}_{x\ge\ell_2(\tau_i^{*-})}\beta(\tau_i^{*-})^2]\\&\qquad+(z_1+q_1(\tau_i^{*+}))[\mathbb{1}_{x\le\ell_1(\tau_i^{*-})}\alpha(\tau_i^{*-})+\mathbb{1}_{x\ge\ell_2(\tau_i^{*-})}\beta(\tau_i^{*-})].\end{aligned}$$

The solution of (4.31a) is given by the following equation: (see Appendix 4.7.2)

$$p_1(t) = \frac{1}{b_x}\left(-a + \frac{\theta}{2} - \frac{\theta}{C_{1,i}e^{\theta t}+1}\right),\ \forall t \in (\tau_i^*, \tau_{i+1}^*),\ i = \{0, 1, \cdots, k\}, \tag{4.32}$$

where

$$\theta = 2\sqrt{a^2 - w_1 b_x}, \tag{4.33}$$

$$C_{1,i} = \begin{cases} \left(\frac{2\theta}{\theta - 2b_x s_1 - 2a} - 1\right) e^{-\theta T} & \text{if } i = k \\ \left(\frac{2\theta}{\theta - 2a} - 1\right) e^{-\theta \tau_i^*} & \text{if } i < k \end{cases}. \tag{4.34}$$

Using the value of $p_1(t)$ given in (4.32), we obtain

$$a_x(t) = a + b_x p_1(t) = \frac{\theta}{2} - \frac{\theta}{C_{1,i}e^{\theta t}+1},\ \forall t \in (\tau_i^*, \tau_{i+1}^*),\ i = \{0, 1, \cdots, k\}.$$

Substituting $a_x(t)$ and $p_1(t)$ in (4.29) yields the equilibrium strategy of Player 1

$$\gamma^*(t, x) = \left(\frac{\theta}{2} - \frac{\theta}{C_{1,i}e^{\theta t}+1}\right) x + b_x q_1(t),\ \forall t \in (\tau_i^*, \tau_{i+1}^*),\ i = \{0, 1, \cdots, k\}. \tag{4.35}$$

4.4.2 Impulse control problem of Player 2

Player 2 solves her impulse control problem for a given equilibrium strategy γ^* of Player 1.

Similar to Bertola et al. (2016), the impulse controls are assumed to be threshold policies which together with the cost structure of Player 2 lead to the following conjecture on the form of the value function of Player 2:

$$V_2(t, x) = \begin{cases} \Phi_2(t, \alpha(t)) + C + c(\alpha(t) - x) & x \leq \ell_1(t) \\ \Phi_2(t, x) & x \in (\ell_1(t), \ell_2(t)) \\ \Phi_2(t, \beta(t)) + D + d(x - \beta(t)) & x \geq \ell_2(t) \end{cases}. \tag{4.36}$$

The value function V_2 coincides with continuous and continuously differentiable function Φ_2 in the impulse free region $\mathcal{C}$. In the intervention region, the value function is equal to the sum of the intervention cost incurred by the player to shift the state to the continuation region and the cost-to-go (that is equal to $\Phi_2(t, \alpha(t)$ or $\Phi_2(t, \beta(t)$ depending on the state value at the impulse time) by playing optimally afterwards. When the state lies in the continuation region, that is, $x \in (\ell_1(t), \ell_2(t))$, the value function of Player 2 satisfies (4.16a) with equality:

$$\frac{\partial \Phi_2(t, x)}{\partial t} + \left(\frac{\partial \Phi_2}{\partial x}\right)(ax + b\gamma^*(t, x)) + \frac{1}{2} w_2 (x - \rho_2)^2 = 0.$$

We conjecture that Φ_2 is quadratic in state in the impulse free region because the cost functions are quadratic in state, and takes the following form:

$$\Phi_2(t,x) = \frac{1}{2}p_2(t)x^2 + q_2(t)x + n_2(t). \tag{4.37}$$

Substituting the partial derivatives of $\Phi_2(t,x)$ and the equilibrium control of Player 1 from (4.29) in the above equation yields

$$\frac{1}{2}\dot{p}_2(t)x^2 + \dot{q}_2(t)x + \dot{n}_2(t) + (p_2(t)x + q_2(t))\,a_x(t)x + b_x q_1(t)\,(p_2(t)x + q_2(t)) \\ + \frac{1}{2}w_2x^2 + \frac{1}{2}w_2\rho_2^2 - w_2x\rho_2 = 0.$$

On comparing the coefficients, we obtain,

$$\dot{p}_2(t) = -w_2 - 2p_2(t)a_x(t),\ p_2(T) = s_2, \tag{4.38a}$$

$$\dot{q}_2(t) = -a_x(t)q_2(t) - b_xp_2(t)q_1(t) + w_2\rho_2,\ q_2(T) = -s_2\rho_2, \tag{4.38b}$$

$$\dot{n}_2(t) = -b_xq_1(t)q_2(t) - \frac{1}{2}w_2\rho_2^2,\ n_2(T) = \frac{1}{2}s_2\rho_2^2. \tag{4.38c}$$

Remark 4.1 *In the impulse control problem studied in Bertola et al. (2016), $p_2(\cdot)$ is assumed to be continuously differentiable in t. In our game problem, $a_x(\cdot)$ depends on the discontinuous function $p_1(\cdot)$ that has jumps due to the interventions by Player 2. Similarly, $q_2(\cdot)$ and $n_2(\cdot)$ are non-differentiable functions*

For the iLQDG, we consider the problem parameters for which the following assumption holds:

Assumption 4.4 *For $t \in [0,T]$, $p_2(t) > 0$.*

The above assumption is satisfied when $w_1 = s_1 = 0$ in which case $a_x(t) = a$ and therefore, ϕ_2 is convex in x for $(t,x) \in \mathcal{C}$.

Intervention set and continuation set

In the intervention region ($(t,x) \in \Sigma\backslash\mathcal{C}$), (4.16b) holds with equality, that is,

$$V_2(t,x) = \mathcal{R}V_2(t,x) = \max_{\eta\in\Omega_v}(V_2(t,x+\eta) + g(\eta)). \tag{4.39}$$

For the problem parameters assumed in this section, V_2 is convex in x (see Assumption 4.4) and continuously differentiable for $x \in \mathcal{C}$. Since $\alpha(t), \beta(t) \in \mathcal{C}$, we can use the first-order conditions to obtain

$$\frac{\partial V_2(t,\alpha(t))}{\partial x} + \frac{\partial g(\eta)}{\partial \eta} = 0, \; x \leq \ell_1(t), \tag{4.40}$$

$$\frac{\partial V_2(t,\beta(t))}{\partial x} + \frac{\partial g(\eta)}{\partial \eta} = 0, \; x \geq \ell_2(t). \tag{4.41}$$

Using the quadratic form of the value function in (4.37) for $x \in \mathcal{C}$, we obtain

$$\begin{aligned} \frac{\partial V_2}{\partial x}(t,\alpha(t)) &= p_2(t)\alpha(t) + q_2(t) = -c, \\ \frac{\partial V_2}{\partial x}(t,\beta(t)) &= p_2(t)\beta(t) + q_2(t) = d. \end{aligned} \tag{4.42}$$

Therefore, the following functions α and β give the state values after an impulse occurs at equilibrium:

$$\alpha(t) = -\frac{q_2(t)+c}{p_2(t)}, \forall t \in [0,T], \tag{4.43a}$$

$$\beta(t) = \frac{d - q_2(t)}{p_2(t)}, \; \forall t \in [0,T]. \tag{4.43b}$$

The functions α and β are continuous in time with possible kinks at the impulse instants.

Since (4.16b) holds with equality in the intervention region, we obtain

$$V_2(t,x) = \begin{cases} V_2(t,\alpha(t)) + C + c(\alpha(t) - x) & x \leq \ell_1(t) \\ V_2(t,\beta(t)) + D + d(x - \beta(t)) & x \geq \ell_2(t) \end{cases}. \tag{4.44}$$

Also, $\alpha(t)$ and $\beta(t)$ lie in the continuation region $\mathcal{C}$ which implies $V_2(t,\alpha(t)) = \Phi_2(t,\alpha(t))$ and $V_2(t,\beta(t)) = \Phi_2(t,\beta(t))$. For $x = \ell_1(t)$ and $x = \ell_2(t)$, we substitute (4.37) in the above equations and simplify to obtain

$$\frac{1}{2}p_2(t)\ell_1(t)^2 + q_2(t)\ell_1(t) = \frac{1}{2}p_2(t)\alpha(t)^2 + q_2(t)\alpha(t) + C + c(\alpha(t) - \ell_1(t)), \tag{4.45a}$$

$$\frac{1}{2}p_2(t)\ell_2(t)^2 + q_2(t)\ell_2(t) = \frac{1}{2}p_2(t)\beta(t)^2 + q_2(t)\beta(t) + D + d(\ell_2(t) - \beta(t)). \tag{4.45b}$$

To characterize the left boundary of the continuation region, we substitute $\alpha(t)$ in (4.45a) to obtain

$$p_2(t)\ell_1(t)^2 + 2(q_2(t)+c)\ell_1(t) - p_2(t)\left(-\frac{q_2(t)+c}{p_2(t)}\right)^2$$

$$- 2(q_2(t) + c)\left(-\frac{q_2(t)+c}{p_2(t)}\right) - 2C = 0$$
$$\Rightarrow p_2(t)\ell_1(t)^2 + 2(q_2(t)+c)\ell_1(t) + \frac{(q_2(t)+c)^2}{p_2(t)} - 2C = 0.$$

Since $C > 0$, $p_2(t) > 0$, and $\ell_1(t) < \alpha(t)$, the left boundary of the continuation region is given by

$$\ell_1(t) = \frac{-c - q_2(t) - \sqrt{2Cp_2(t)}}{p_2(t)}. \tag{4.46a}$$

On substituting $\beta(t)$ in (4.45b), we obtain the right boundary of the continuation region

$$p_2(t)\ell_2(t)^2 + 2(q_2(t)-d)\ell_2(t) - p_2(t)\left(\frac{d-q_2(t)}{p_2(t)}\right)^2$$
$$- 2(q_2(t)-d)\left(\frac{d-q_2(t)}{p_2(t)}\right) - 2D = 0$$
$$\Rightarrow p_2(t)\ell_2(t)^2 + 2(q_2(t)-d)\ell_2(t) + \frac{(d-q_2(t))^2}{p_2(t)} - 2D = 0.$$

From $D > 0, p_2(t) > 0$ and $\ell_2(t) > \beta(t)$, we obtain

$$\ell_2(t) = \frac{-q_2(t) + d + \sqrt{2Dp_2(t)}}{p_2(t)}. \tag{4.46b}$$

Remark 4.2 *In Bertola et al. (2016), the authors analytically characterized $\alpha(t)$, $\beta(t)$, $\ell_1(t)$, $\ell_2(t)$ for their impulse optimal control problem. However, in iLQDG, these functions are coupled with Player 1's problem. As a result, we obtain a semi-analytic characterization of these variables in terms of the problem parameters.*

By construction, $V_1(t,x)$ satisfies the sufficient conditions in (4.14a)-(4.14c) and therefore, V_1 is a value function of Player 1. In the next theorem, we give conditions for which $V_2(t,x)$ in (4.36) satisfies the QVIs (4.16a-4.16d) to conclude that $V_1(t,x)$ and $V_2(t,x)$ are indeed the value functions of the players.

Theorem 4.2 *$V_2(t,x)$in* (4.36) *is the value function of Player 2 if $\ell_1(t) \leq x_{11}(t)$ and $\ell_2(t) \geq x_{22}(t)$ for each $t \in [0,T]$ where $\ell_1(t)$ and $\ell_2(t)$ are given in* (4.46a) *and* (4.46b), *respectively,*

$$x_{11}(t) = \frac{(ca + w_2\rho_2) - \sqrt{\theta_\alpha(t)}}{w_2}, \tag{4.47a}$$

$$x_{22}(t) = \frac{-(da - w_2\rho_2) + \sqrt{\theta_\beta(t)}}{w_2}, \tag{4.47b}$$

$$\theta_\alpha(t) = c^2a^2 + 2w_2\left(ca\rho_2 - \frac{\partial\Phi_2(t,\alpha(t))}{\partial t}\right), \tag{4.47c}$$

$$\theta_\beta(t) = d^2a^2 - 2w_2\left(da\rho_2 - \frac{\partial\Phi_2(t,\beta(t))}{\partial t}\right), \tag{4.47d}$$

and $x_{11}(t)$ and $x_{22}(t)$ are well-defined with $\theta_\alpha(t) \geq 0$ and $\theta_\beta(t) \geq 0$ for all $t \in [0,T]$.

Proof. From (4.42), we have $\frac{\partial V_2(t,\alpha(t))}{\partial x} = -c$ and $\frac{\partial V_2(t,\beta(t))}{\partial x} = d$. Using the convexity of V_2 in x for $x \in \mathcal{C}$ (Assumption 4.4), we obtain

$$-c < \frac{\partial V_2(t,x)}{\partial x} < d,\ x \in (\alpha(t), \beta(t)).$$

Therefore, $\mathcal{R}V_2(t,x) = \min(C,D)$ for $x \in (\alpha(t),\beta(t))$.
When $x \in (\ell_1(t), \alpha(t))$, we have $\frac{\partial V_2(t,x)}{\partial x} < -c$ and for $x \in (\beta(t), \ell_2(t))$, we obtain $\frac{\partial V_2(t,x)}{\partial x} > d$ from the convexity of $V_2(t,x)$ in $x \in (\ell_1(t), \ell_2(t)$. Therefore, the operator $\mathcal{R}$ satisfies the following system

$$\mathcal{R}V_2(t,x) = \begin{cases} \Phi_2(t,\alpha(t)) + C + c(\alpha(t) - x) & x \leq \alpha(t) \\ \Phi_2(t,x) + \min(C,D) & x \in (\alpha(t),\beta(t)) \\ \Phi_2(t,\beta(t)) + D + d(x - \beta(t)) & x \geq \beta(t) \end{cases}. \tag{4.48}$$

Clearly, $V_2 - \mathcal{R}V_2 < 0$ in the continuation region and $V_2(t,x) = \mathcal{R}V_2(t,x)$ in the intervention region.
Next, we derive the conditions under which the value function of Player 2 satisfies (4.16a). For $x < \ell_1(t)$, we have

$$V_2(t,x) = \Phi_2(t,\alpha(t)) + C + c(\alpha(t) - x). \tag{4.49}$$

When $x < \ell_1(t)$, we obtain

$$\begin{aligned} &\frac{\partial V_2(t,x)}{\partial t} + \mathcal{H}_2(x, \gamma^*(t,x), \frac{\partial V_2(t,x)}{\partial x}) \\ &= \frac{\partial V_2(t,x)}{\partial t} + \frac{1}{2}w_2(x-\rho_2)^2 + \frac{\partial V_2(t,x)}{\partial t}(ax + \mathbb{1}_{\ell_1(t)<x<\ell_2(t)}\, b\gamma^*(t,x)) \\ &= \frac{\partial \Phi_2(t,\alpha(t))}{\partial t} + \left(\frac{\partial \Phi_2(t,\alpha(t))}{\partial x} + c\right)\frac{d\alpha(t)}{dt} - ca + \frac{1}{2}w_2x^2 + \frac{1}{2}w_2\rho_2^2 - w_2x\rho_2. \end{aligned}$$

Substituting (4.42) in the above equation, we get the roots of the above equation as follows:

$$x_{11}, x_{12} = \frac{(ca + w_2\rho_2) \pm \sqrt{\theta_\alpha(t)}}{w_2}, \tag{4.50}$$

where $x_{11} < x_{12}$, and $\theta_\alpha(t)$ is given by equation (4.47c). Therefore, (4.16a) holds if $\ell_1(t) \leq x_{11}(t)$ and $\theta_\alpha(t) \geq 0$ for all $t \in [0, T]$.
For $x > \ell_2(t)$, we obtain

$$\begin{aligned}
&\frac{\partial V_2(t,x)}{\partial t} + \mathcal{H}_2(x, \gamma^*(t,x), \frac{\partial V_2(t,x)}{\partial x}) \\
&= \frac{\partial V_2(t,x)}{\partial t} + \frac{1}{2}w_2(x-\rho_2)^2 + \frac{\partial V_2(t,x)}{\partial t}(ax + \mathbb{1}_{\ell_1(t)<x<\ell_2(t)} b\gamma^*(t,x)) \\
&= \frac{\partial \Phi_2(t,\beta(t))}{\partial t} + \left(\frac{\partial \Phi_2(t,\beta(t))}{\partial x} - d\right)\frac{d\beta(t)}{dt} + da + \frac{1}{2}w_2 x^2 + \frac{1}{2}w_2\rho_2^2 - w_2 x\rho_2.
\end{aligned}$$

On substituting (4.42) in the above equation, we obtain the roots of the above equation as follows:

$$x_{21}, x_{22} = \frac{-(da - w_2\rho_2) \pm \sqrt{\theta_\beta(t)}}{w_2}, \tag{4.51}$$

where $x_{21}(t) < x_{22}(t)$ and $\theta_\beta(t)$ is given by (4.47d). Therefore, (4.16a) holds if $\ell_2(t) \geq x_{22}(t)$. ■

Since analytical solutions cannot be obtained for iLQDG, we need numerical methods to characterize the equilibrium.

4.5 Numerical examples

To illustrate our results, we consider two iLQDGs with the problem parameters given in Table 4.1. To determine the impulse instants, we formulate a constrained non-linear optimization problem (see Sadana et al., 2021) and numerically compute the solution using the `fmincon` solver in MATLAB.

Figure 4.1 corresponds to the iLQDG with problem parameters in the first row of Table 4.1. In this case, an equilibrium strategy of Player 2 is to give an impulse only at the initial time and shift the state from the initial value of 6, which is in the intervention region, to $\beta(0)$ in the continuation region. Clearly, the state lies within the boundaries of the continuation region from $t = 0^+$ to $t = T$ and the

T	a	b	w_1	w_{11}	s_1	r_1	z_1	w_2	s_2	c	C	D	d	ρ_1	ρ_2	x_0
1	0.15	−0.18	1	0	1	0.6	0	5	1	8	1	10	1	8	4	6
1	0.1	−0.5	0	2	0	2	3	0.2	3	8	0.1	10	8	10	6	14

Table 4.1 – Parameters for numerical example

QVIs hold. Hence, it is not optimal for Player 2 to give additional impulses during the game. The best response of Player 2 to an equilibrium strategy of Player 1 is to give at least one impulse because the state $x(t)$ remains in the intervention region if Player 2 does not given any impulse as can be seen in Figure 4.1.

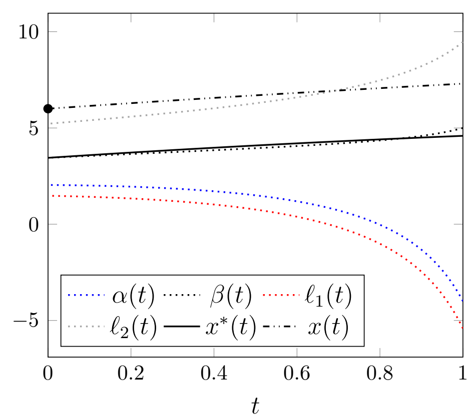


Figure 4.1 – Evolution of intervention region and state variable for parameters in the first row of Table 4.1. $x^*(t)$ denotes the equilibrium state trajectory and $x(t)$ denotes the state evolution when no impulse occurs.

Next, we consider the problem parameters in the second row of Table 4.1 where Player 1's payoff varies linearly with state. As shown in Figure 4.2, the equilibrium strategy of Player 2 is to intervene twice during the game by giving impulses at $\tau_1^* = 0$ and at $\tau_2^* = 0.88$. In this case, we can show that the QVIs hold. The functions α, β, ℓ_1 and ℓ_2 have kinks at the impulse instant $\tau_2^* = 0.88$ due to jumps in the equilibrium control of Player 1.

4.6 Conclusions

In this paper, we have considered a two-player finite-horizon nonzero-sum differential game where Player 1 uses piecewise-continuous controls and Player 2 uses impulse controls. We determined an upper bound on the number of impulses and

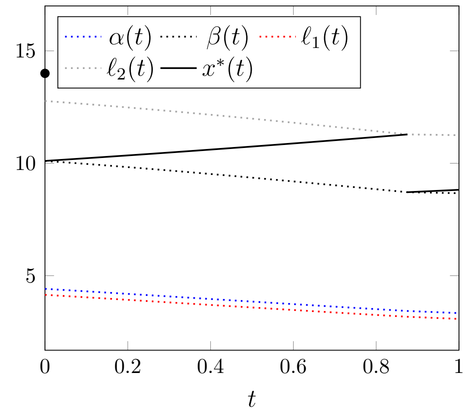


Figure 4.2 – Evolution of intervention region and state variable for parameters in the second row of Table 4.1

provided sufficient conditions to characterize the feedback Nash equilibrium for this general class of differential games with impulse controls. The sufficient conditions are given as a coupled system of Hamilton-Jacobi-Bellman equations with jumps and quasi-variational inequalities. To the best of our knowledge, this is the first characterization of feedback Nash equilibrium in differential games with impulse controls where at least one player uses piecewise-continuous controls. This also contrasts our work with earlier papers on impulse games where equilibrium solutions were derived for problems where both players use impulse controls only. Furthermore, we extended a well-studied linear-quadratic impulse control problem to a game setting with both players using their controls to minimize the cost associated with the state deviating from their target values.

4.7 Appendix

4.7.1 Proof of Proposition 4.1

A feasible strategy of Player 2 is not to give any impulse in $[0, T]$ so that

$$\sum_{j=i}^{k} \mathbb{1}_{t \leq \tau_j \leq T}\, b_2(x(\tau_j), v_i) = 0, \tag{4.52}$$

and it follows from the boundedness of h_2 and s_2 in Assumption 4.1 that

$$V_2(t, x) \leq \int_t^T h_2(x(s), \gamma^*(s, x(s)))ds + s_2(x(T)) \leq c_{h_2}(T - t) + c_{s_2}.$$

Next, for any $\epsilon > 0$, take a strategy $\delta_{[t,T]} \in \Delta_{[t,T]}$ so that

$$V_2(t,x) + \epsilon > J_2(x, \gamma^*_{[t,T]}, \delta_{[t,T]}) \geq -c_{h_2}(T-t) - c_{s_2}.$$

This proves that the value function is bounded such that

$$|V_2(t,x)| \leq c_{h_2}(T-t) + c_{s_2},\ \forall (t,x) \in [0,T] \times \mathbb{R}^n. \tag{4.53}$$

For any $\epsilon > 0$, consider a ϵ optimal strategy v^ϵ with $N(v^\epsilon)$ impulses. From the boundedness of h_2, we obtain

$$V_2(t,x) + \epsilon > J_2(x, \gamma_1, v^\epsilon) \geq -c_{h_2}(T-t) + \mu N(v^\epsilon) - c_{s_2}.$$

Using the above relation and (4.53), we obtain

$$-c_{h_2}(T-t) + \mu N(v^\epsilon) - c_{s_2} < c_{h_2}(T-t) + c_{s_2} + \epsilon.$$

Since $\mu > 0$, we can rewrite the above inequality as follows:

$$N(v^\epsilon) < \frac{2\left(c_{h_2}(T-t) + c_{s_2}\right) + \epsilon}{\mu}.$$

For a feasible strategy of Player 1 given by $\gamma(t,x) = 0$ for all $(t,x) \in \Sigma$ and the upper bound K on the number of impulses, we have

$$\begin{aligned} V_1(t,x) &\leq \int_t^T h_1(x(s),0)ds + \sum_{j=i}^{k} \mathbb{1}_{t \leq \tau_j < T}\, b_1(x(\tau_i), v_j) + s_1(x(T)) \\ &\leq c_{h_1}(T-t) + \sum_{j=i}^{k} \mathbb{1}_{t \leq \tau_j < T}\, b_1(x(\tau_i), v_j) + s_1(x(T)) \\ &\leq c_{h_1}(T-t) + K c_{b_1} + c_{s_1}. \end{aligned}$$

where the last inequality follows from the boundedness of b_1 and s_1 in Assumption 4.1. For any $\epsilon > 0$, we take a strategy $\gamma_{[t,T]} \in \Gamma_{[t,T]}$ so that

$$V_1(t,x) + \epsilon > J_1(x, \gamma_{[t,T]}, \delta^*_{[t,T]}) \geq -c_{h_1}(T-t) - K c_{b_1} - c_{s_1}.$$

This proves that the value function of Player 1 is bounded.

4.7.2 Analytical solution of ODE

To solve the differential equation $\dot{p_1}(t) + b_x(p_1(t))^2 + 2ap_1(t) + w_1 = 0$ for $t \in (\tau_i, \tau_{i+1})$,
$i \in \{0, 1, \cdots, k\}$, we substitute $p_1(t) = \frac{\dot{\mu}(t)}{b_x\mu(t)}$ to obtain a second-order ordinary differential equation $\ddot{\mu}(t) + 2a\dot{\mu}(t) + b_x w_1 \mu(t) = 0$. When $\theta = 2\sqrt{a^2 - w_1 b_x}$, the solution of this equation is

$$\mu(t) = e^{-at}(F_1 e^{\frac{1}{2}\theta t} + F_2 e^{-\frac{1}{2}\theta t})$$

where F_1 and F_2 are constants. So, $p_1(t)$ is given by

$$\begin{aligned} p_1(t) = \frac{\dot{\mu}(t)}{b_x\mu(t)} &= \frac{-a\mu(t) + \frac{\theta}{2}e^{-at}(F_1 e^{\frac{1}{2}\theta t} - F_2 e^{-\frac{1}{2}\theta t})}{b_x e^{-at}(F_1 e^{\frac{1}{2}\theta t} + F_2 e^{\frac{1}{2}\theta t})} \\ &= \frac{1}{b_x}\left(-a + \frac{\theta}{2} - \frac{\theta}{C_1 e^{\theta t} + 1}\right) \end{aligned}$$

Substitute $p_1(T) = s_1$ in the above equation to obtain

$$C_{1,k} = \left(\frac{2\theta}{\theta - 2b_x s_1 - 2a} - 1\right) e^{-\theta T}. \tag{4.54}$$

For $p_1(\tau_i) = 0,\ i < k$, we obtain

$$C_{1,i} = \left(\frac{2\theta}{\theta - 2a} - 1\right) e^{-\theta \tau_i}. \tag{4.55}$$

References

Azimzadeh, P. (2019). A zero-sum stochastic differential game with impulses, precommitment, and unrestricted cost functions. *Applied Mathematics & Optimization*, 79(2):483–514.

Aïd, R., Basei, M., Callegaro, G., Campi, L., and Vargiolu, T. (2020). Nonzero-sum stochastic differential games with impulse controls: A verification theorem with applications. *Mathematics of Operations Research*, 45(1):205–232.

Başar, T. and Olsder, G. J. (1999). *Dynamic Noncooperative Game Theory, 2nd edition*. Society for Industrial and Applied Mathematics, Philadelphia, PA.

Başar, T. (2018). Time consistency and robustness of equilibria in noncooperative dynamic games. In der Ploeg, F. V. and de Zeeuw, A., editors, *Dynamic Policy Games in Economics*, pages 9 – 54. Springer International Publishing, North-Holland.

Başar, T., Haurie, A., and Zaccour, G. (2018). Nonzero-Sum Differential Games. In Başar, T. and Zaccour, G., editors, *Handbook of Dynamic Game Theory*, pages 61–110. Springer International Publishing, Cham.

Basei, M., Cao, H., and Guo, X. (2019). Nonzero-sum stochastic games and mean-field games with impulse controls.

Bensoussan, A., Liu, R. H., and Sethi, S. P. (2005). Optimality of an (s, s) policy with compound poisson and diffusion demands: A quasi-variational inequalities approach. *SIAM Journal on Control and Optimization*, 44(5):1650–1676.

Bertola, G., Runggaldier, W. J., and Yasuda, K. (2016). On classical and restricted impulse stochastic control for the exchange rate. *Applied Mathematics & Optimization*, 74(2):423–454.

Cadenillas, A. and Zapatero, F. (1999). Optimal central bank intervention in the foreign exchange market. *Journal of Economic Theory*, 87(1):218 – 242.

Campi, L. and De Santis, D. (2020). Nonzero-sum stochastic differential games between an impulse controller and a stopper. *Journal of Optimization Theory and Applications*, 186(2):688–724.

Chahim, M. (2013). *Impulse control maximum principle: Theory and applications*. PhD thesis, Tilburg University.

Chahim, M., Grass, D., Hartl, R. F., and Kort, P. M. (2017). Product innovation with lumpy investment. *Central European Journal of Operations Research*, 25(1):159–182.

Chang, D., Wang, H., and Wu, Z. (2013). Maximum principle for non-zero sum differential games of BSDEs involving impulse controls. In *Proceedings of the 32nd Chinese Control Conference*, pages 1564–1569.

Chikrii, A. A., Matychyn, I. I., and Chikrii, K. A. (2007). Differential games with impulse control. In Jørgensen, S., Quincampoix, M., and Vincent, T. L., editors, *Advances in Dynamic Game Theory. Annals of the International Society of Dynamic Games*, volume 9, pages 37–55. Birkhäuser, Boston.

Cosso, A. (2013). Stochastic Differential Games Involving Impulse Controls and Double-Obstacle Quasi-variational Inequalities. *SIAM Journal on Control and Optimization*, 51(3):2102–2131.

Dockner, E. J., Jorgensen, S., Long, N. V., and Sorger, G. (2000). *Differential Games in Economics and Management Science*. Cambridge University Press.

El Asri, B. and Mazid, S. (2018). Zero-sum stochastic differential game in finite horizon involving impulse controls. *Applied Mathematics & Optimization*. ePub ahead of print, September 24.

Engwerda, J. C. (2005). *LQ Dynamic Optimization and Differential Games*. Wiley, New York.

Erdlenbruch, K., Jean-Marie, A., Moreaux, M., and Tidball, M. (2013). Optimality of impulse harvesting policies. *Economic Theory*, 52(2):429 – 459.

Ferrari, G. and Koch, T. (2019). On a strategic model of pollution control. *Annals of Operations Research*, 275(2):297–319.

Grass, D. and Chahim, M. (2012). Numerical algorithm for impulse control models. Working paper.

Haurie, A., Krawczyk, J. B., and Zaccour, G. (2012). *Games and Dynamic Games*. World Scientific, Singapore.

Hou, S. H. and Wong, K. H. (2011). Optimal Impulsive Control Problem with Application to Human Immunodeficiency Virus Treatment. *Journal of Optimization Theory and Applications*, 151(2):385–401.

Jeanblanc-Picqué, M. (1993). Impulse control method and exchange rate. *Mathematical Finance*, 3(2):161–177.

Jørgensen, S. and Zaccour, G. (2003). A differential game of retailer promotions. *Automatica*, 39(7):1145 – 1155.

Korn, R. (1998). Portfolio optimisation with strictly positive transaction costs and impulse control. *Finance and Stochastics*, 2(2):85–114.

Leander, R., Lenhart, S., and Protopopescu, V. (2015). Optimal control of continuous systems with impulse controls. *Optimal Control Applications and Methods*, 36(4):535–549.

Reddy, P. V., Wrzaczek, S., and Zaccour, G. (2016). Quality effects in different advertising models - An impulse control approach. *European Journal of Operational Research*, 255(3):984–995.

Runggaldier, W. J. and Yasuda, K. (2018). Classical and restricted impulse control for the exchange rate under a stochastic trend model. *Journal of Economic Dynamics and Control*, 91:369–390. Special Issue in Honour of Prof. Carl Chiarella.

Sadana, U., Reddy, P. V., Başar, T., and Zaccour, G. (2020a). Sampled-data Nash equilibria in differential games with impulse control. Les Cahiers du GERAD G-2020-81 GERAD, HEC Montreal, Canada.

Sadana, U., Reddy, P. V., and Zaccour, G. (2020b). Open-loop and feedback Nash equilibria in scalar linear-state differential games with impulse control. Les Cahiers du GERAD G-2020-19 GERAD, HEC Montreal, Canada.

Sadana, U., Reddy, P. V., and Zaccour, G. (2021). Nash equilibria in nonzero-sum differential games with impulse control. Accepted by European Journal of Operational Research.

Sulem, A. (1986). A solvable one-dimensional model of a diffusion inventory system. *Mathematics of Operations Research*, 11(1):125–133.

Taynitskiy, V., Gubar, E., and Zhu, Q. (2019). Optimal Impulse Control of SIR Epidemics Over Scale-Free Networks. In Song, J. B., Li, H., and Coupechoux, M., editors, *Game Theory for Networking Applications*, pages 119–129. Springer International Publishing, Cham.

Yong, J. (1994). Zero-sum differential games involving impulse controls. *Applied Mathematics and Optimization*, 29(3):243–261.

Zhang, F. (2011). Stochastic differential games involving impulse controls. *ESAIM: Control, Optimisation and Calculus of Variations*, 17(3):749–760.

Conclusion

Motivated by problems in regulation, counter-terrorism and cybersecurity, we study nonzero-sum differential games between two players, one of whom continuously controls the state while the other player intervenes at certain discrete time instants to shift the state value from one level to another. In this class of games with each player using a different kind of control, we characterize open-loop, feedback and sampled-data Nash equilibrium. For linear-quadratic differential games, we show that equilibrium impulse timing can be obtained by solving a constrained non-linear optimization problem. Furthermore, we provide closed-form solutions for equilibrium number, timing and levels of impulses for scalar linear-state differential games with impulse controls. We recover the classical result in linear-state differential games that open-loop and feedback Nash equilibrium coincide when the impulse instants are exogenously given. However, when the impulse player also determines the timing of impulses, we show that open-loop and feedback Nash equilibria do not coincide.

Our work can be extended in several directions. A challenging problem is to study a differential game where both players have both continuous and impulse controls. To explore this research direction, one could consider the scalar linear-state game model that we have studied in the first essay and as a starting point, consider that the impulse player has both piecewise-continuous and impulse controls. Another interesting research direction is to formulate a Stackelberg game model with a leader that uses impulse controls while the multiple followers with their piecewise-continuous controls play a Nash game among themselves. This problem has applications in epidemic control by governments that aim to determine the timing and intensity of lockdowns while the population manage their social interactions keeping in view the loss in their utility due to the enforcement of a lockdown. It is not hard to see that the equilibrium conditions in this game can be obtained using our approach in the first essay with the difference that there will be a co-state equation for each follower in the game. However, the increase in the co-state equations could result in computational difficulties in analyzing these games.

The computation of feedback Nash equilibrium in differential games with impulse controls entails solving the system of quasivariational inequalities, which is a difficult problem. Therefore, for linear-quadratic differential games, we have made regularity assumptions on the value function of the impulse player. For the future, our objective is to develop policy iteration algorithms that can numerically compute the feedback Nash equilibria for a general class of nonzero-sum differential games with impulse controls.