What's hidden in the tails? Revealing and reducing optimistic bias in entropic risk estimation and optimization

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(joint work with Erick Delage and Angelos Georghiou)

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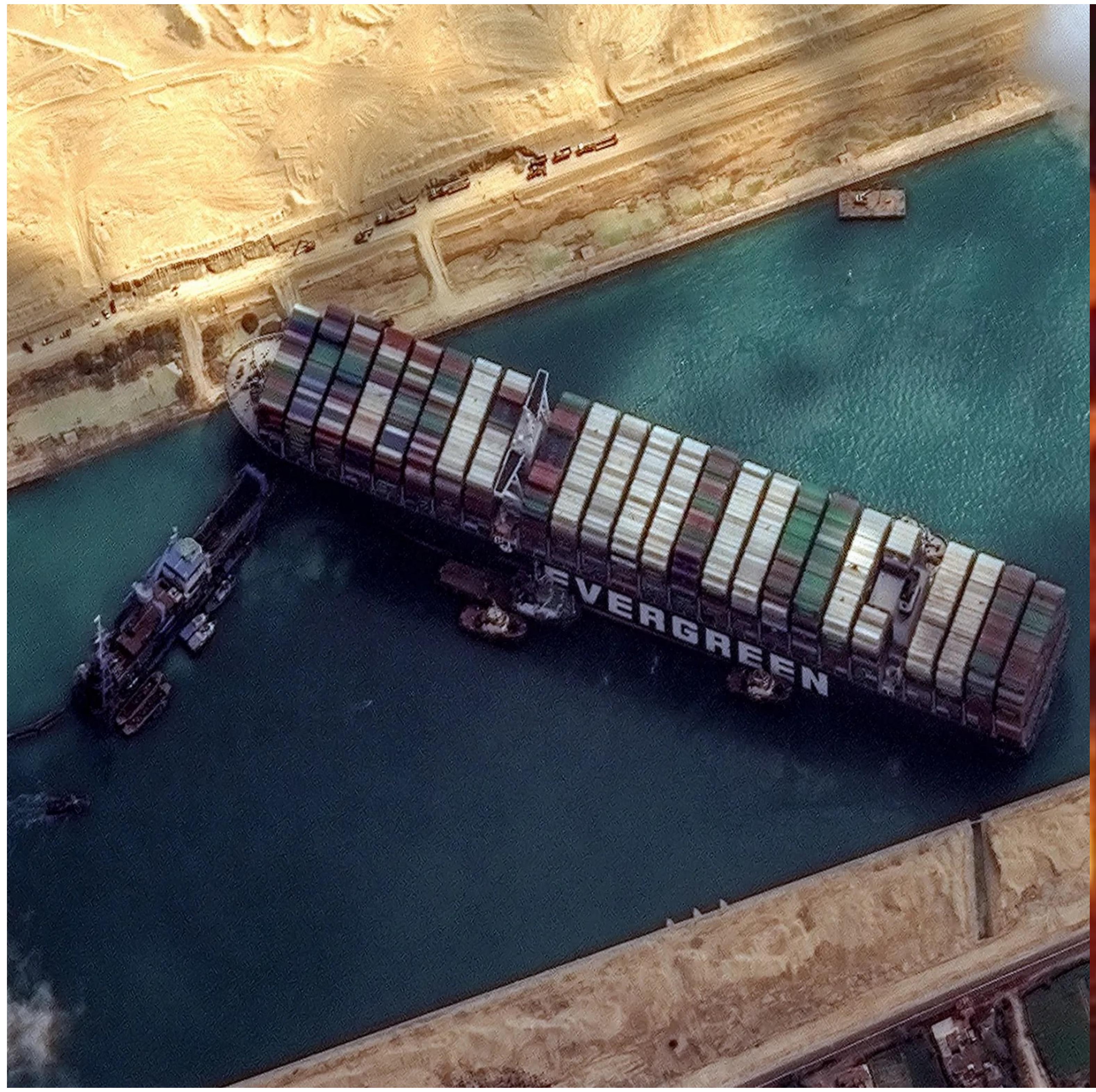
What's hidden in the tails?



Tails and Bias correction

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- Loss is uncertain
- Risk measure maps loss to a real number
- Entropic risk measure accounts for
    - mean
    - variance
    - Higher moments

- Estimation of entropic risk:
  - True risk - Use known loss distribution
  - We use data to construct risk estimator

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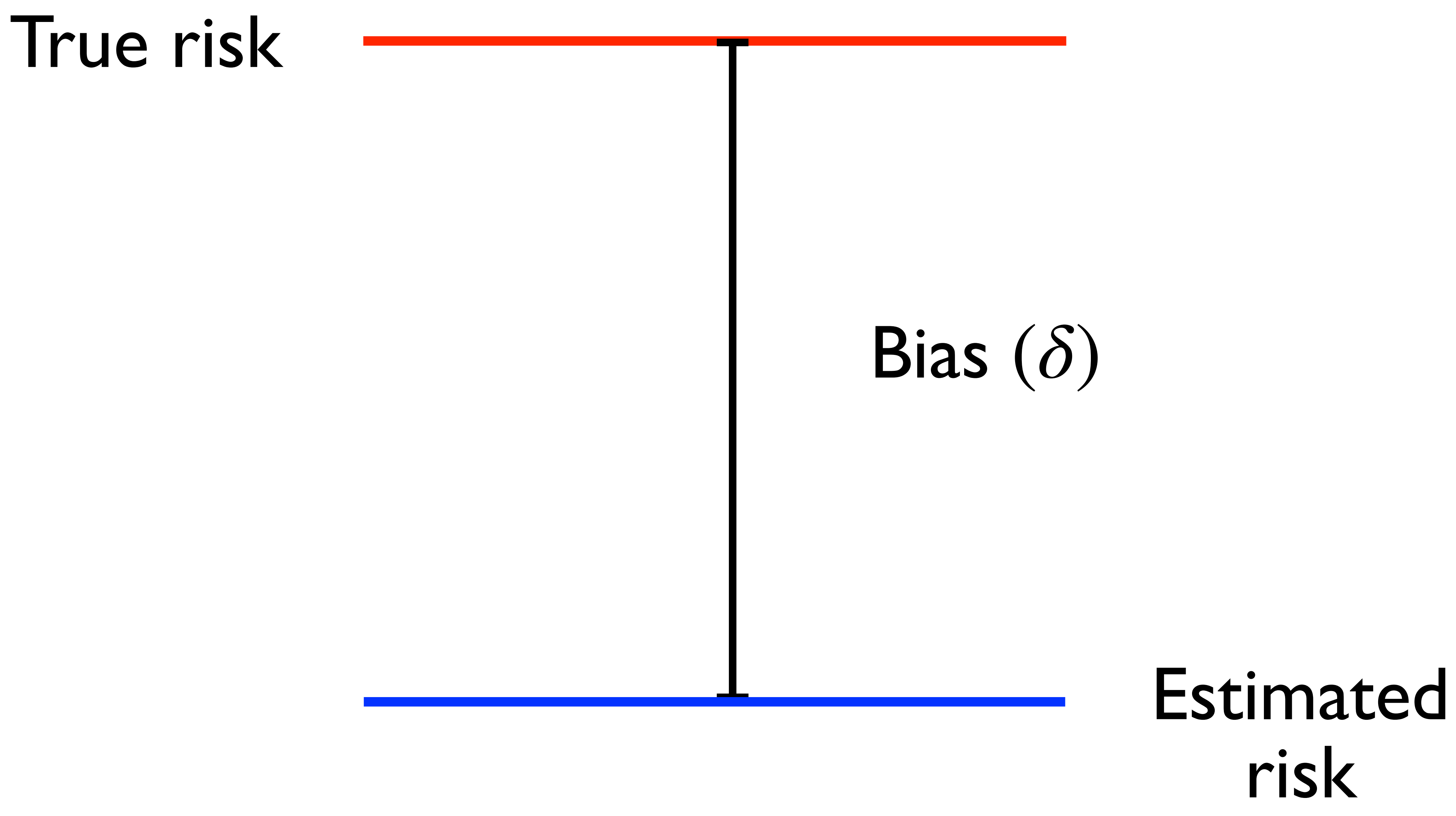

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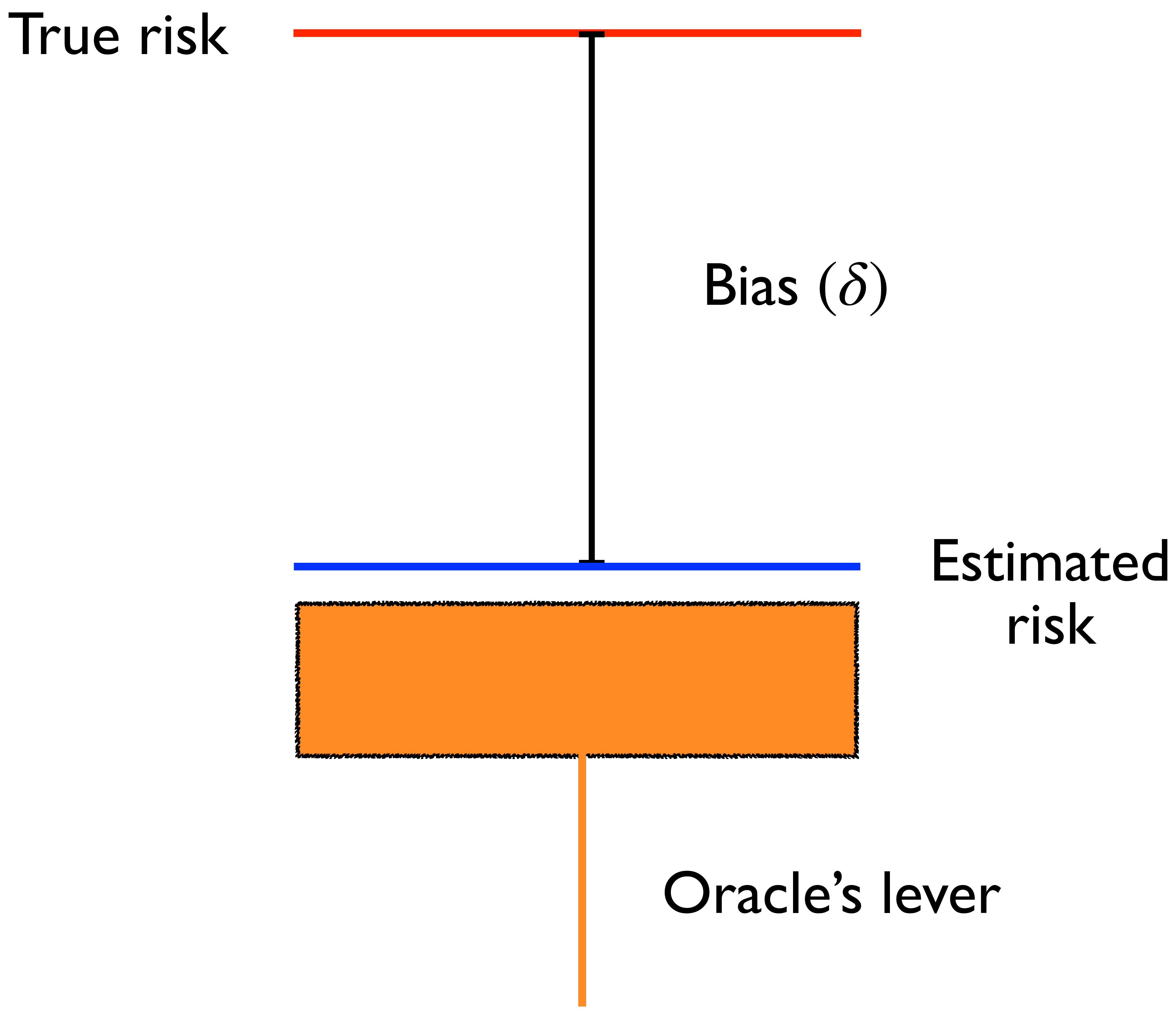

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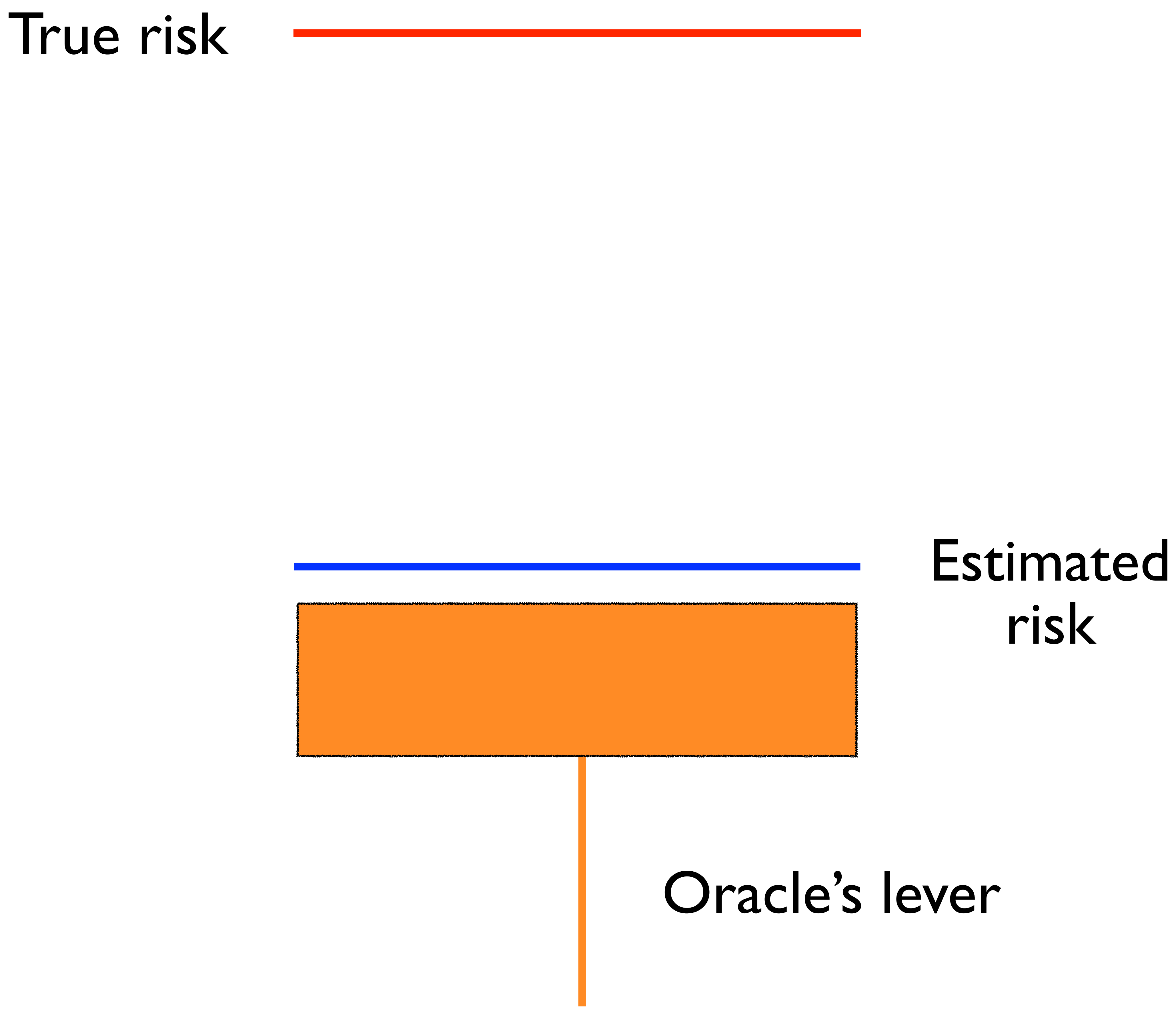

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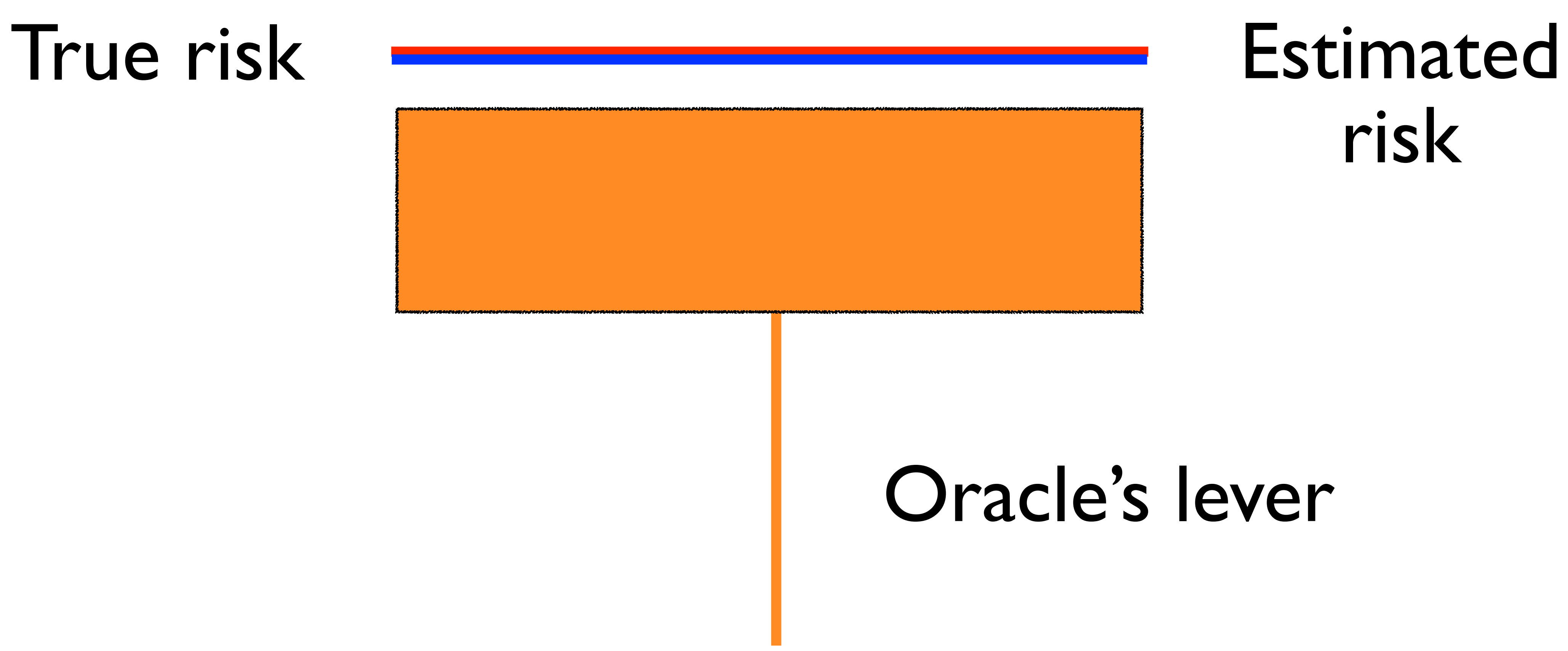

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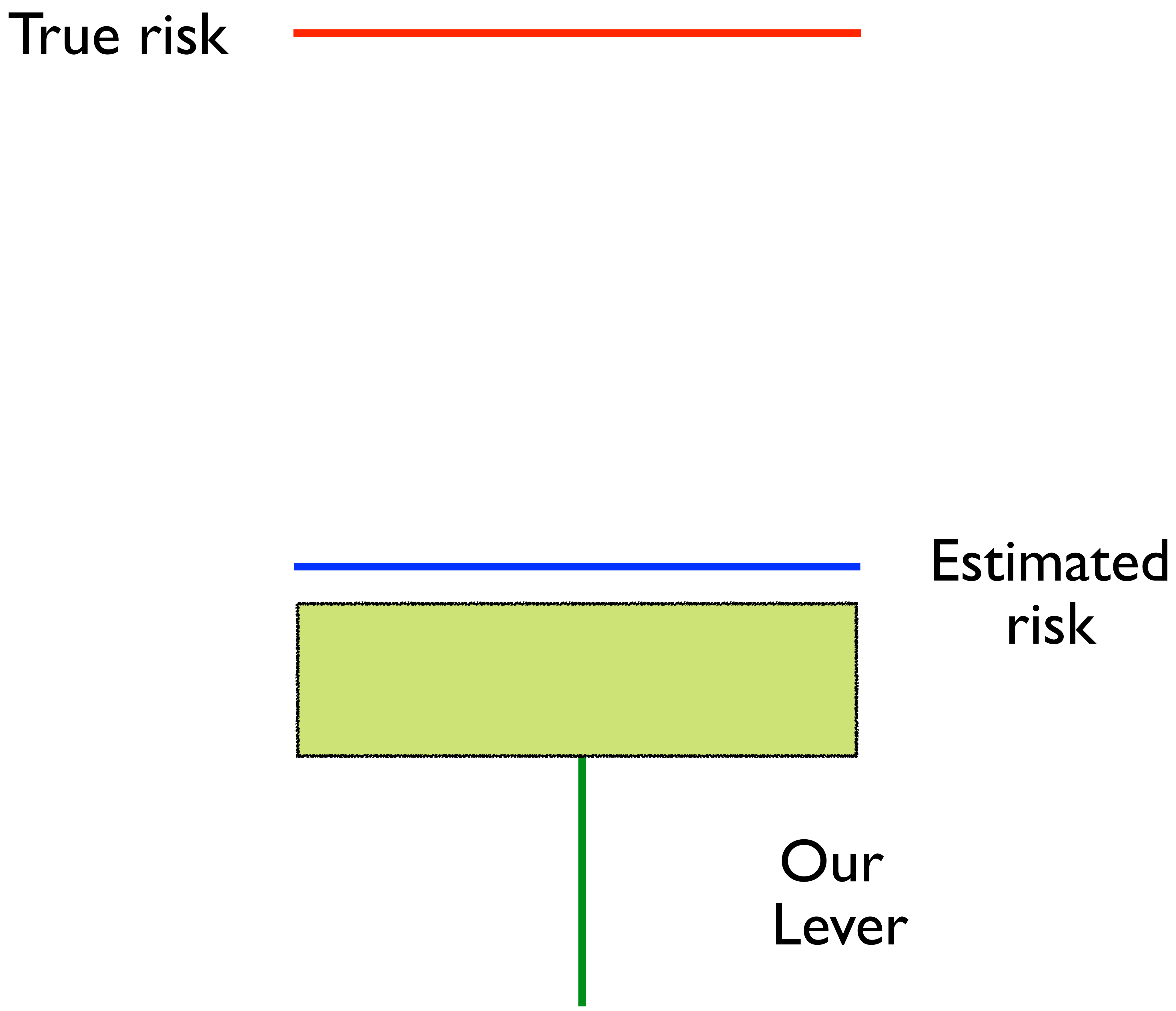

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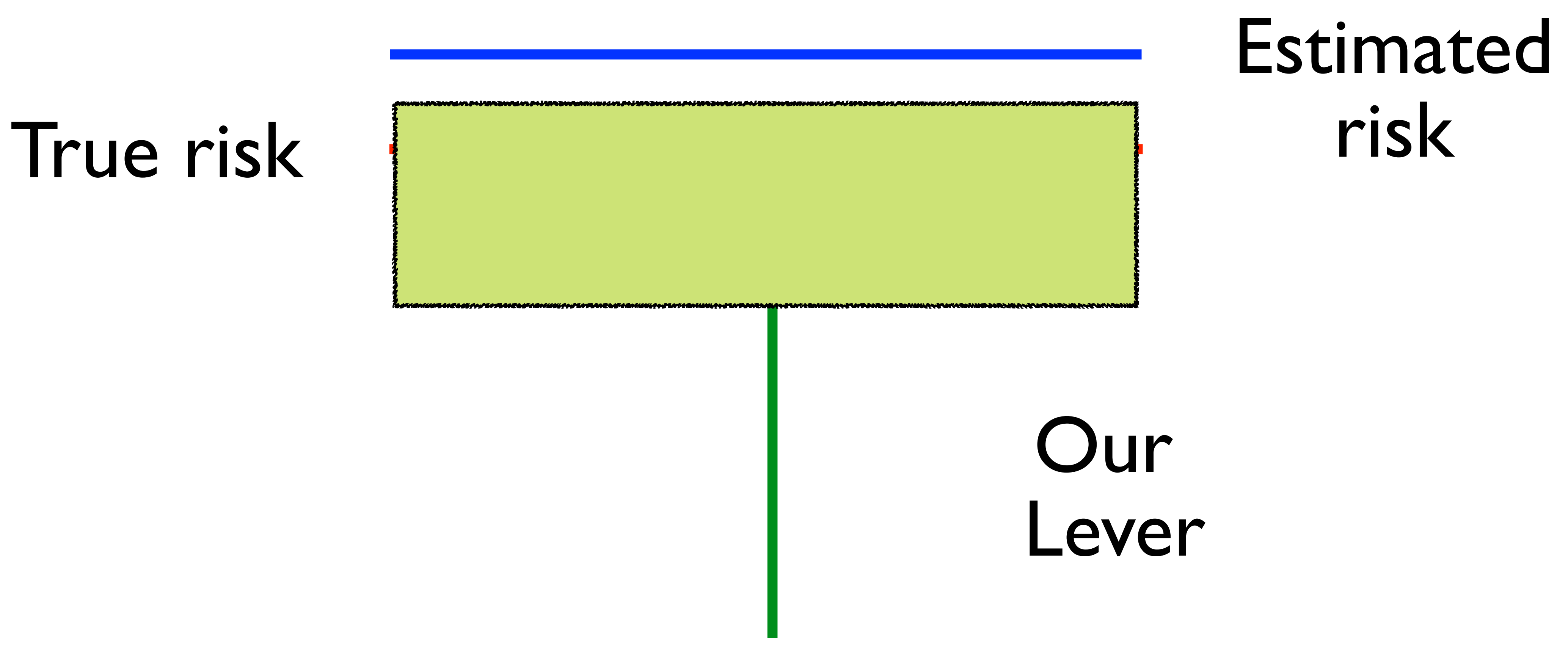

Beyond risk neutrality

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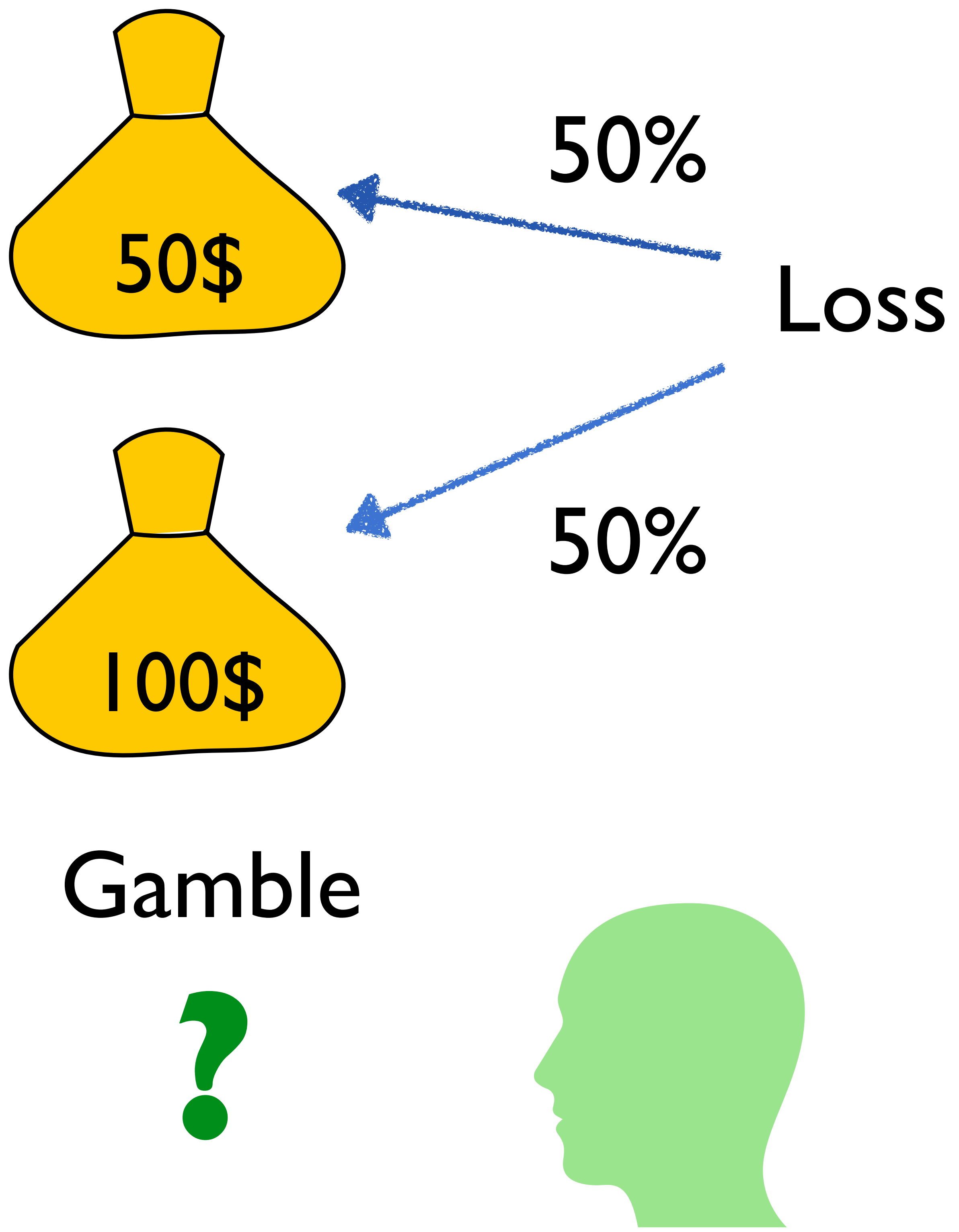

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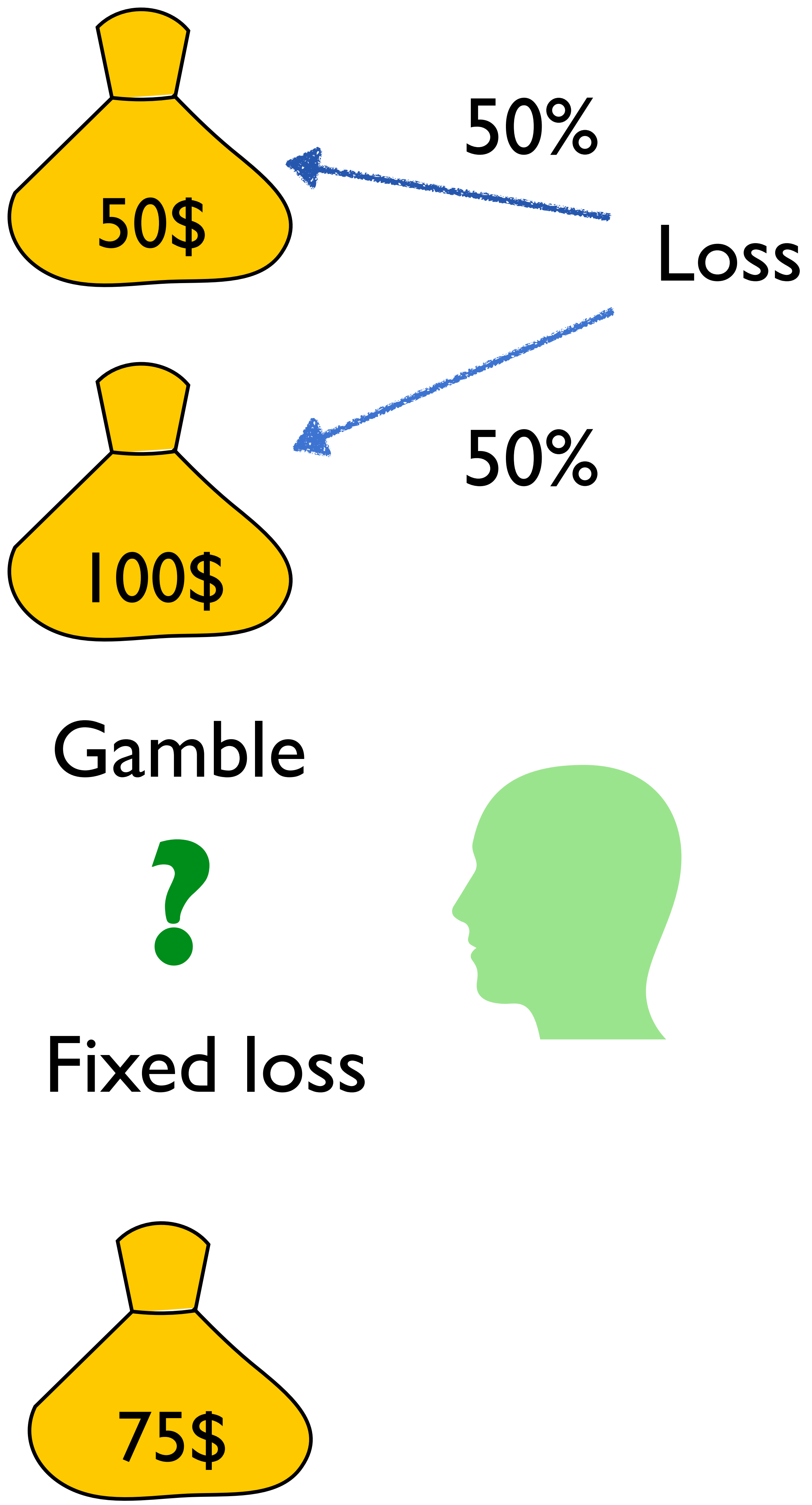
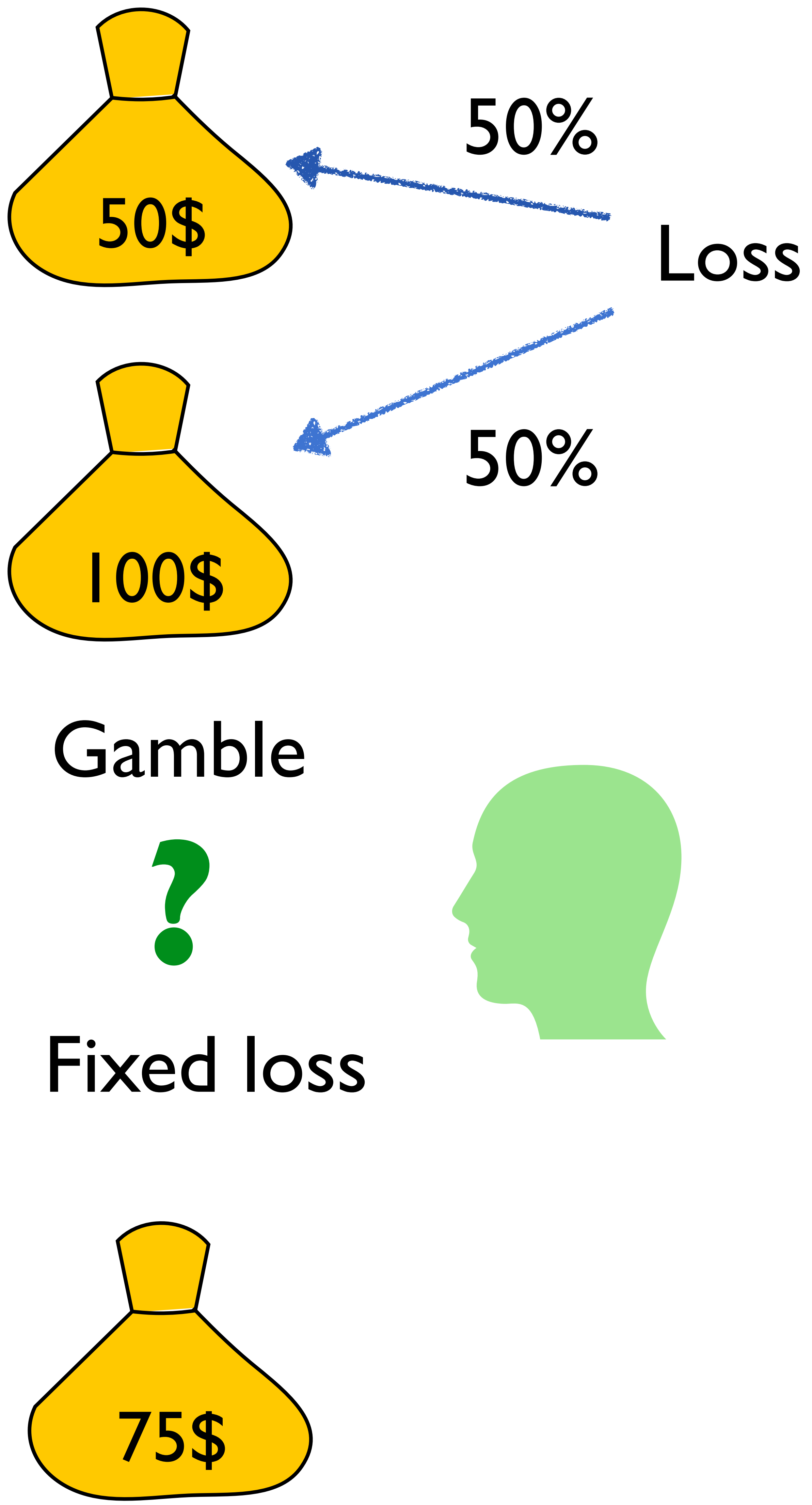


Beyond risk neutrality

Indifference between the two options

- [x] Risk neutral
- [ ] Experiments
- [ ] Entropic risk measure



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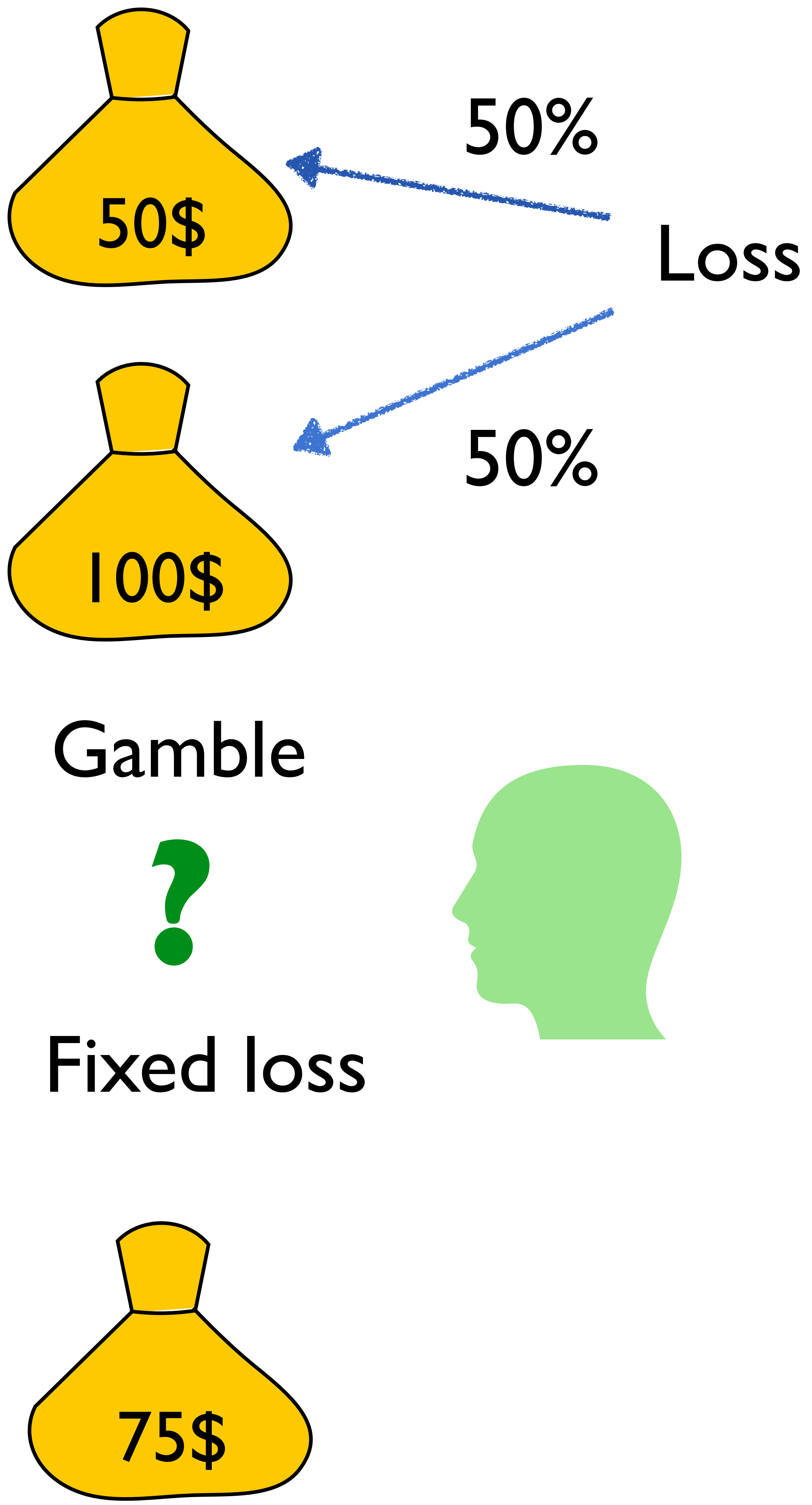

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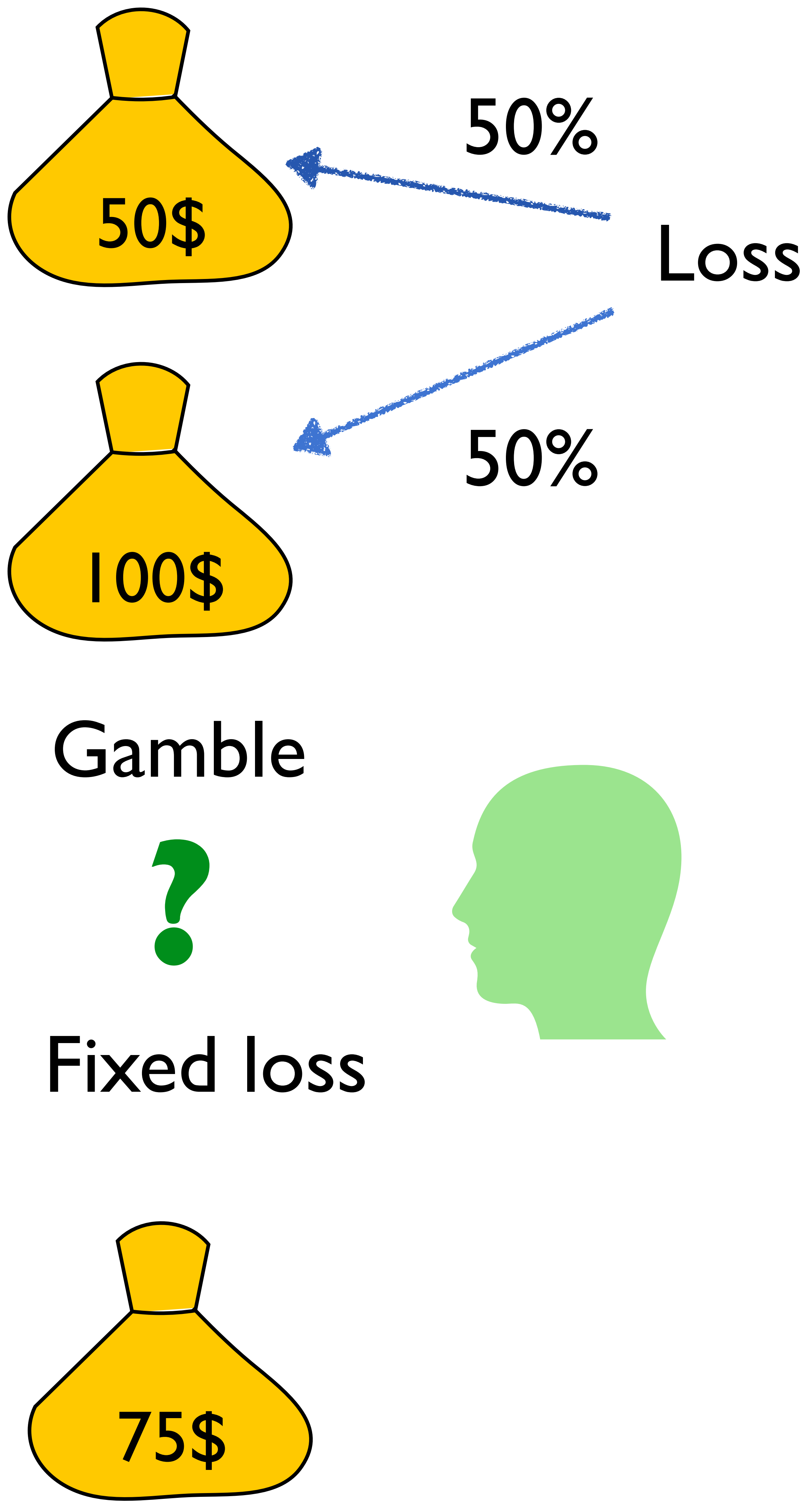


Entropic risk measure

Uncertainty: $\boldsymbol{\xi} \sim \mathbb{P}$ → Loss: $\ell(\boldsymbol{\xi})$ →

Entropic risk

$$\rho_{\mathbb{P}}(\ell(\boldsymbol{\xi})) := \frac{1}{\alpha} \log \left(\mathbb{E}_{\mathbb{P}} \left[\exp(\alpha \ell(\boldsymbol{\xi})) \right] \right)$$

- α is the decision maker's risk aversion
- $\mathbb{P}$ is not known

Entropic risk measure

Data ($\mathscr{D}_N$)

$\boldsymbol{\xi}_1, \boldsymbol{\xi}_2, \ldots, \boldsymbol{\xi}_N$ →

Loss

$\ell(\boldsymbol{\xi}_1), \ell(\boldsymbol{\xi}_2), \ldots, \ell(\boldsymbol{\xi}_N)$ →

Empirical entropic risk

$$\rho_{\hat{\mathbb{P}}_N}(\ell(\boldsymbol{\xi})) := \frac{1}{\alpha} \log \left(\frac{1}{N} \sum_{i=1}^{N} \exp(\alpha \ell(\boldsymbol{\xi}_i)) \right)$$

Empirical entropic risk underestimates true entropic risk:

✓ Jensen's inequality: $\mathbb{E}$[empirical risk] $<$ True risk

✓ Optimized certainty equivalent (OCE) measure

$$\rho_{\mathbb{P}}(\ell(\boldsymbol{\xi})) = \inf_t \mathbb{E}_{\mathbb{P}} \left(t + \frac{1}{\alpha} \exp(\alpha(\ell(\boldsymbol{\xi}) - t)) - \frac{1}{\alpha} \right)$$

replace with $\hat{\mathbb{P}}_N$ (optimizer's curse)

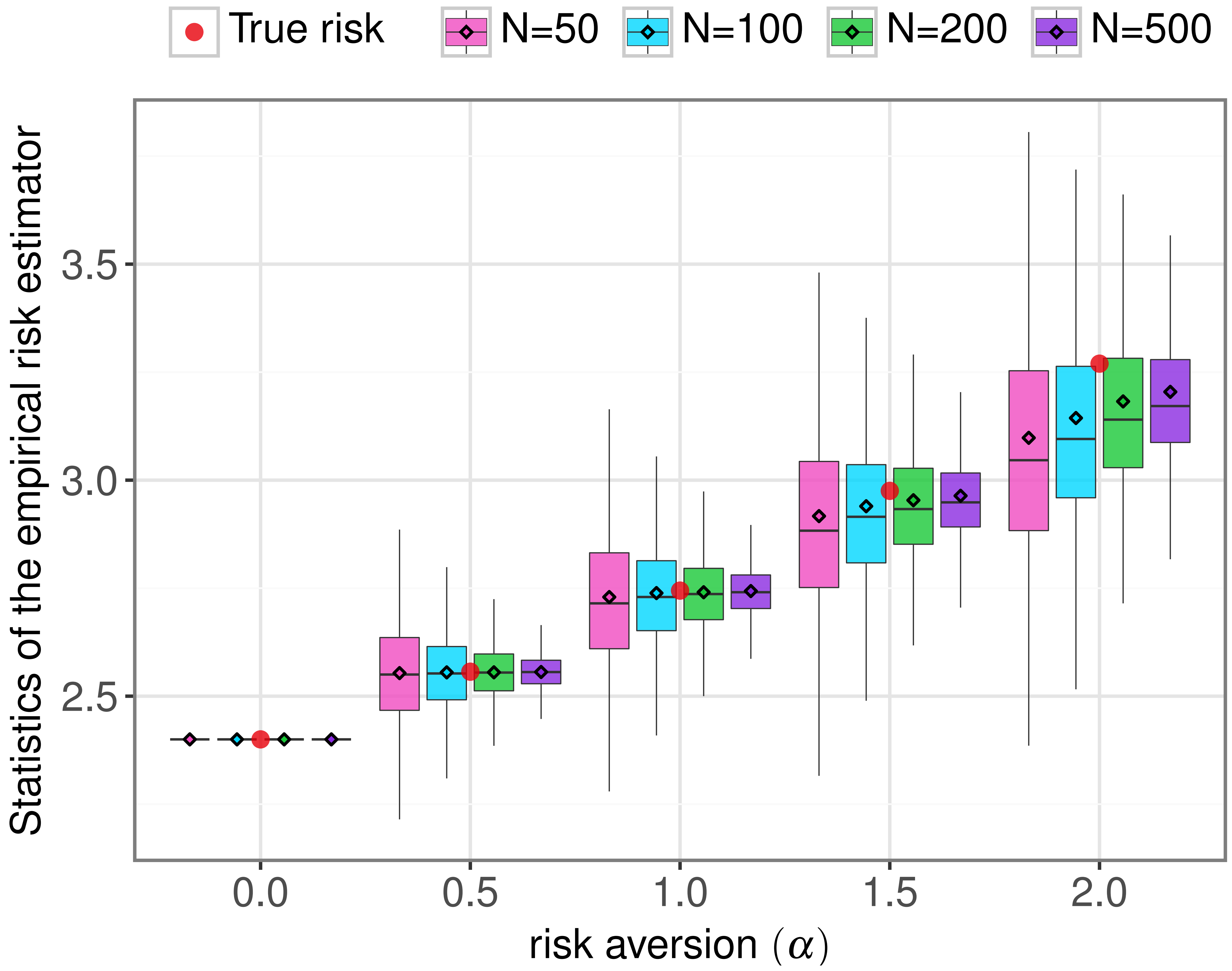
Ex 1: pricing insurance

- Loss $\xi \sim \Gamma(10, 0.24)$
- **Insurer covers the risk:**

$$\text{Premium} = \frac{1}{\alpha} \log \left(\mathbb{E}_{\mathbb{P}} \left[\exp(\alpha \ell(\boldsymbol{\xi})) \right] \right)$$

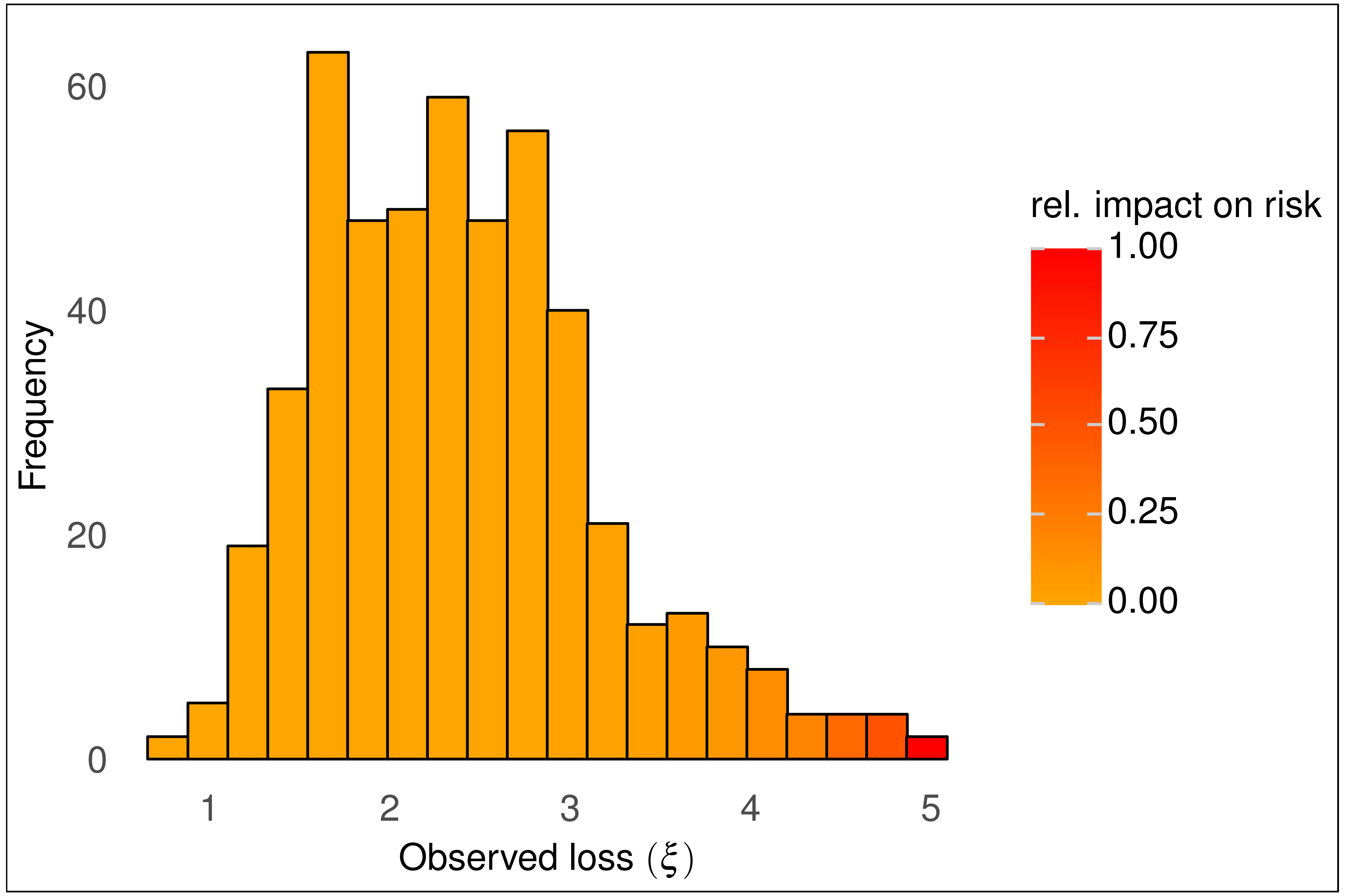
- Sample mean $\to$ true mean slowly:

Gaussian $\xi \implies \exp(\alpha\xi)$ is log-normal



Influence function (IF)

Influence function (IF) - impact of data removal on risk



Bias mitigation with bootstrapping

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Bias: $\delta = \mathbb{E}[\rho_{\mathbb{P}}(\zeta) - \rho_{\hat{\mathbb{P}}_N}(\zeta)]$

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$\rho_{\hat{\mathbb{P}}_N}(\zeta) + \delta_N(\mathbb{Q})$ **almost surely converges to true entropic risk**

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Bias mitigation with bootstrapping

Efficiently computable risk under $\mathbb{Q}$
Gaussian mixture models are universal function approximators

$$\rho_{\mathbb{Q}}(\zeta) = (1/\alpha)\log\left(\sum_y \pi_y \exp(\alpha\mu_y + \alpha^2\sigma_y^2/2)\right)$$

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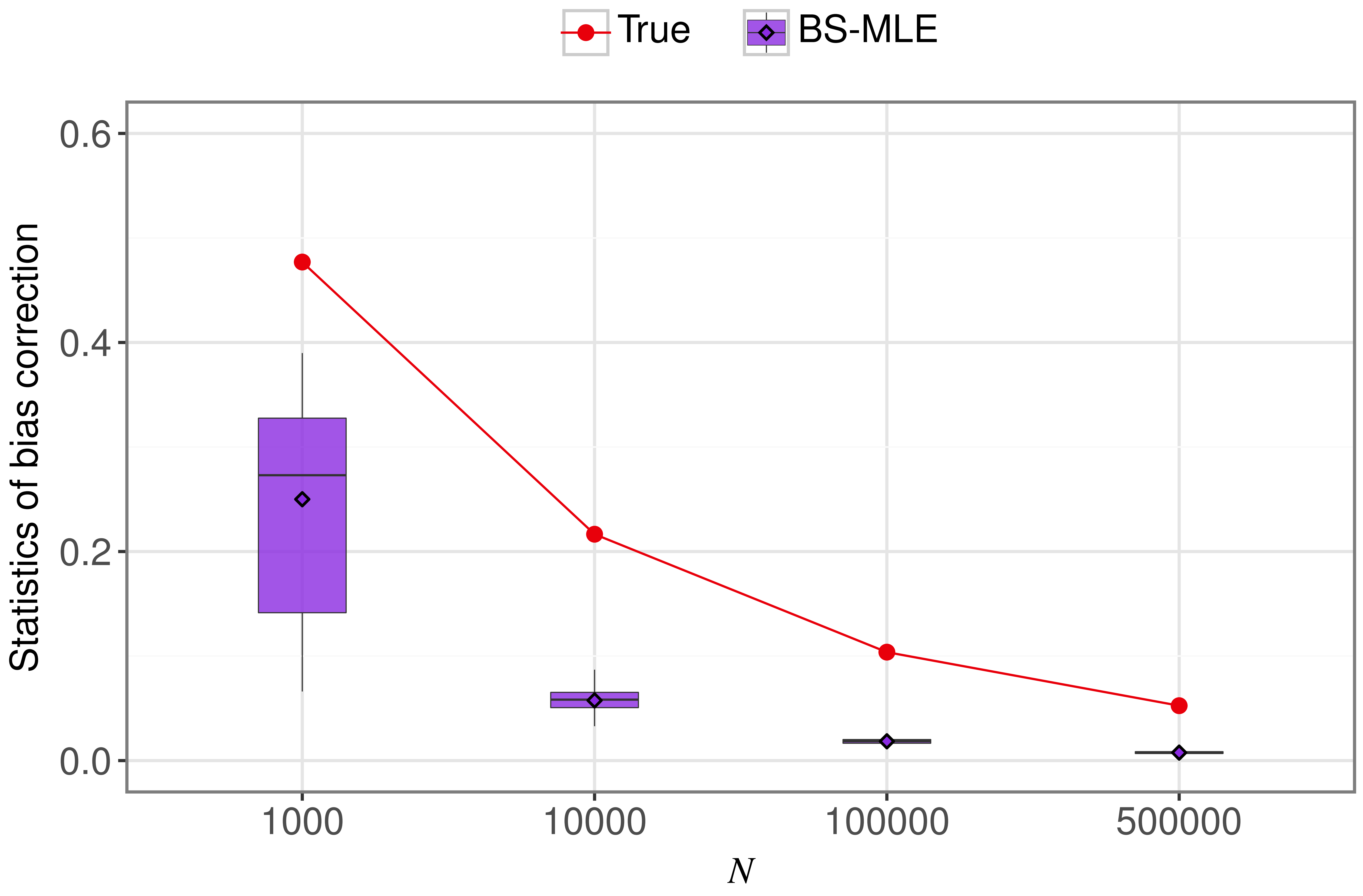
Model 1: Fit using maximum likelihood (BS-MLE)



- Ex: Compute entropic risk
- $\xi \sim \text{GMM}(\pi, \mu, \sigma), \pi = [0.7\ 0.3]$, $\mu = [0.5\ 1], \sigma = [2\ 1]$
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- **Underestimation persists**

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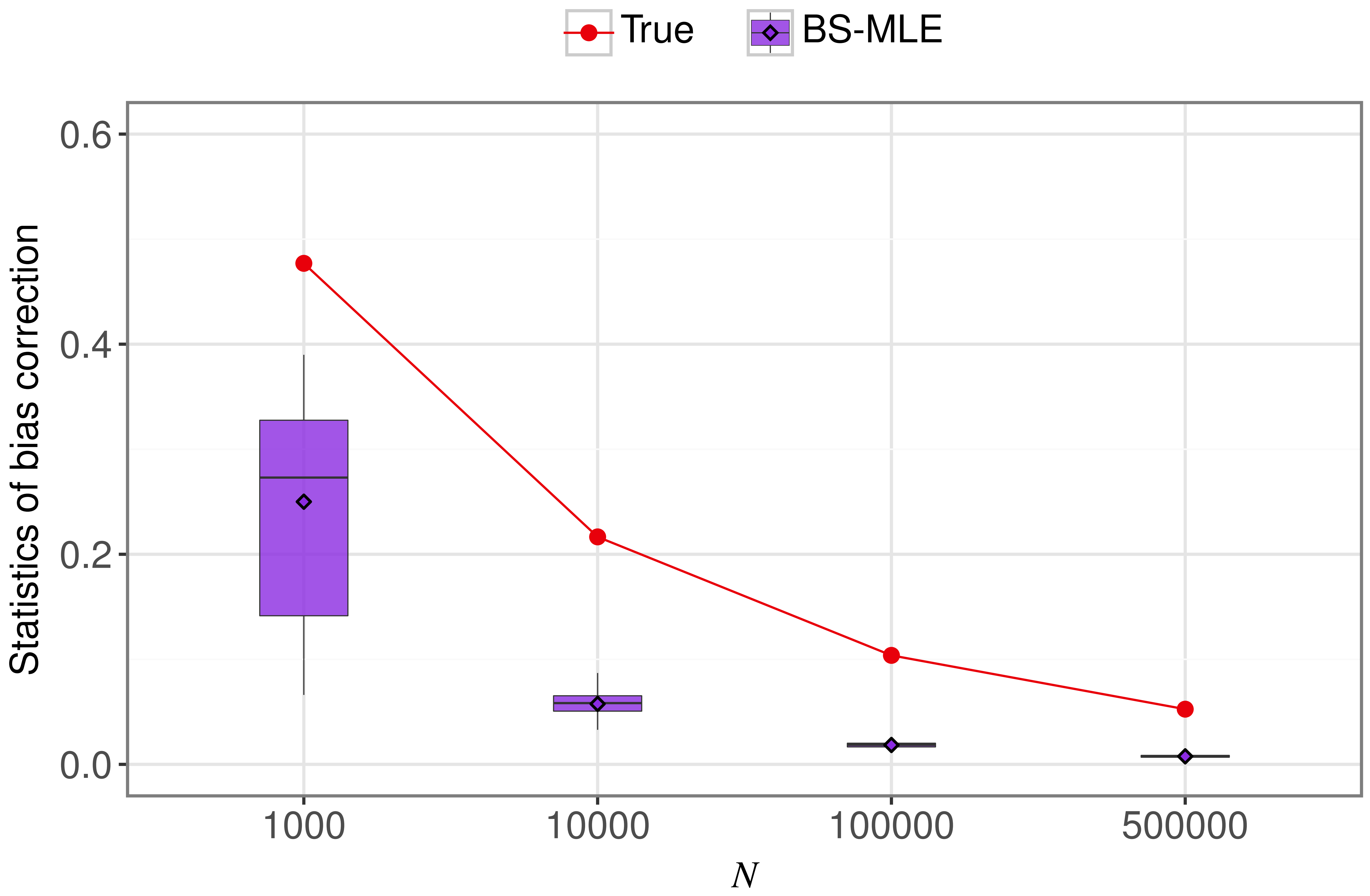

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Bias mitigation using Bias-aware bootstrapping

Wish:

Fit distribution $\mathbb{Q}$ whose samples have the same bias as the bias in the data

Bias mitigation using Bias-aware bootstrapping

Model 2: Entropic risk matching (BS-Match)

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Idea: Match distributions of the entropic risk over the samples

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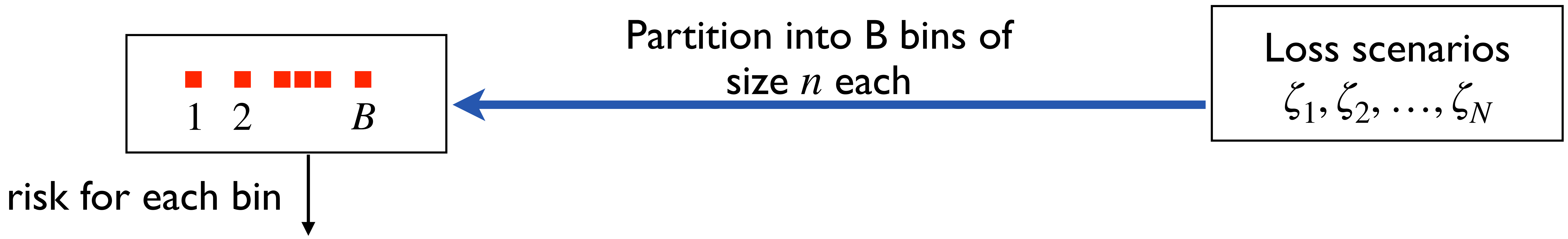

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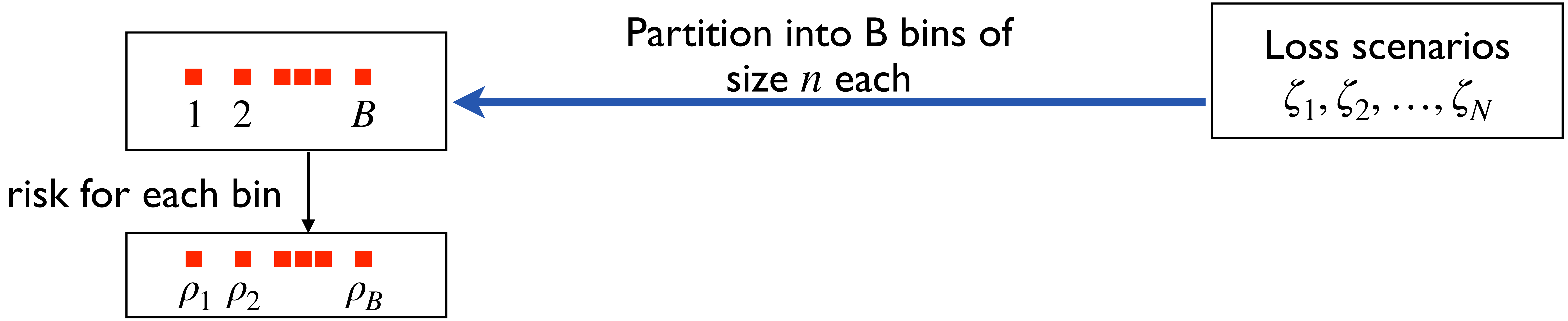

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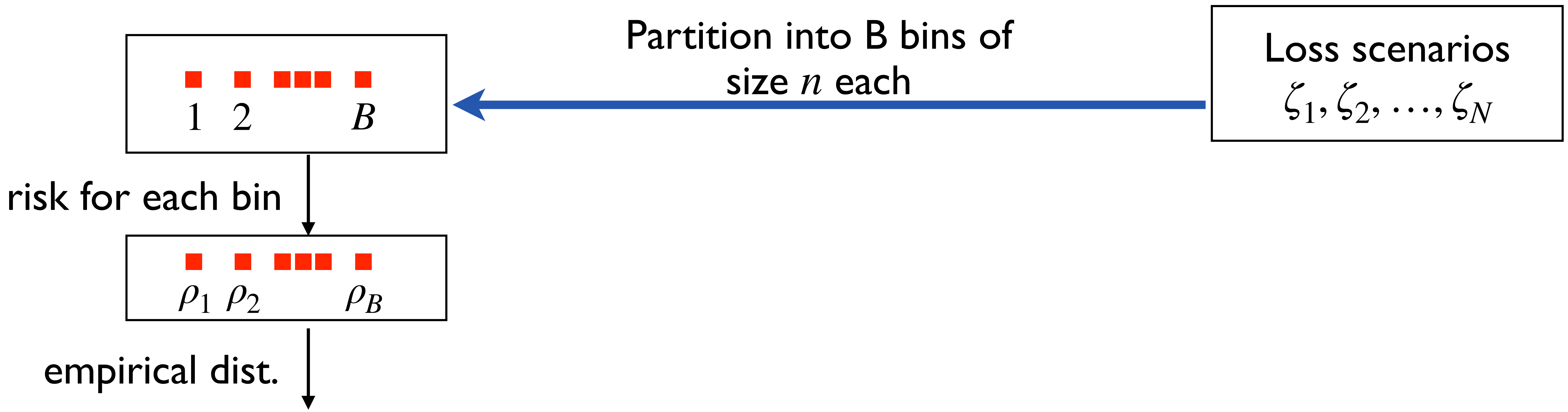

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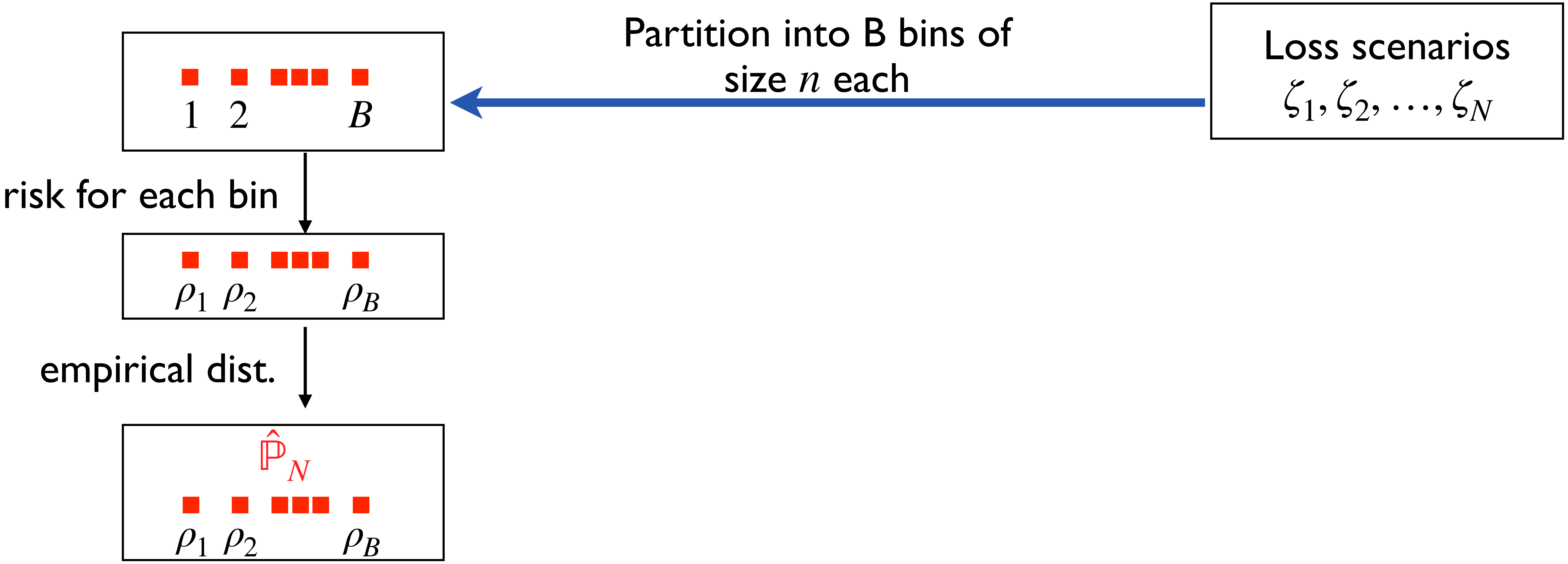

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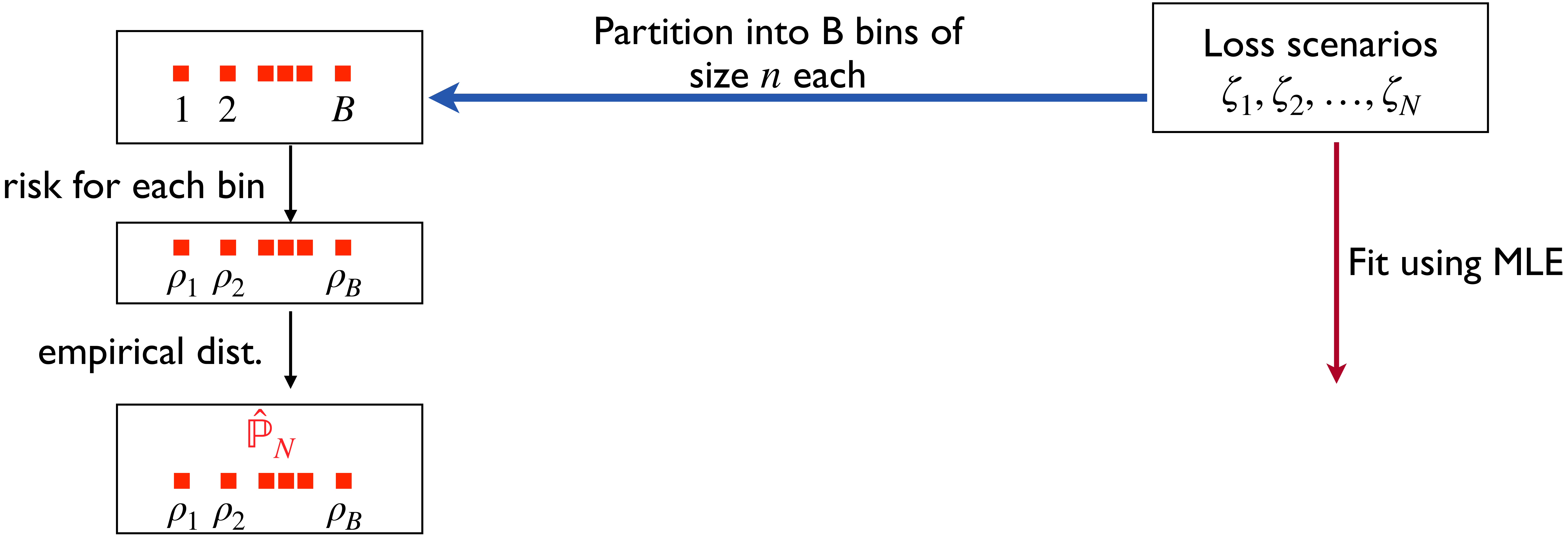

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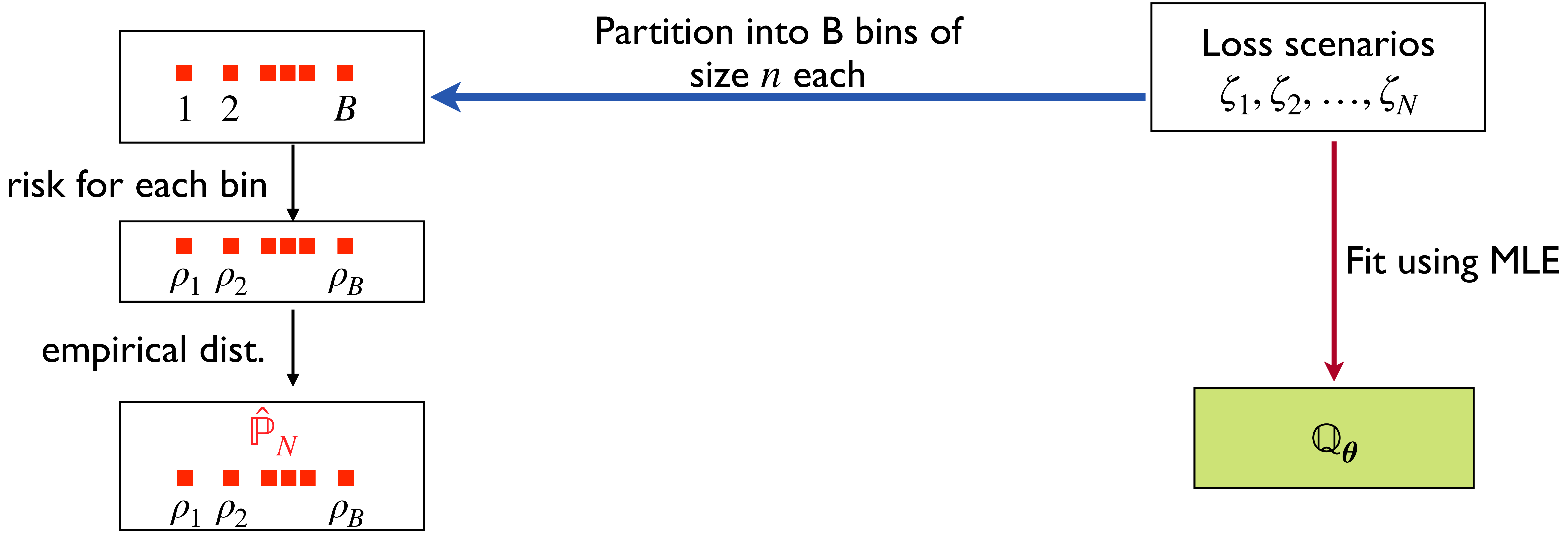

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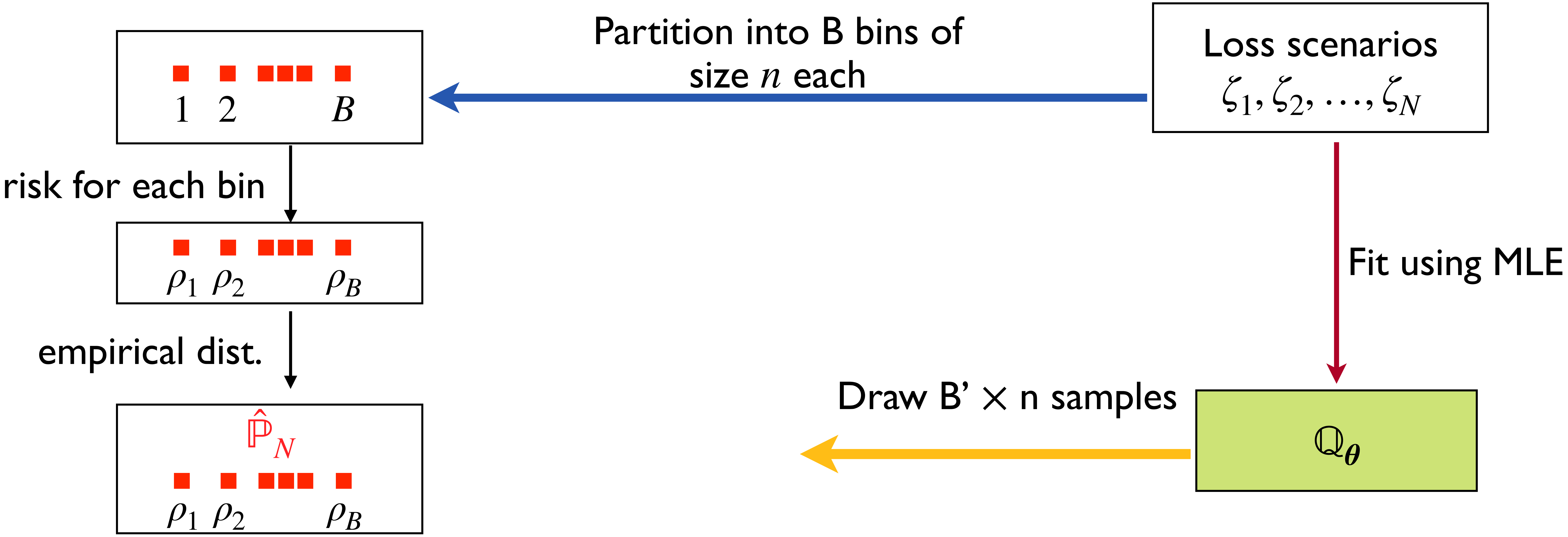

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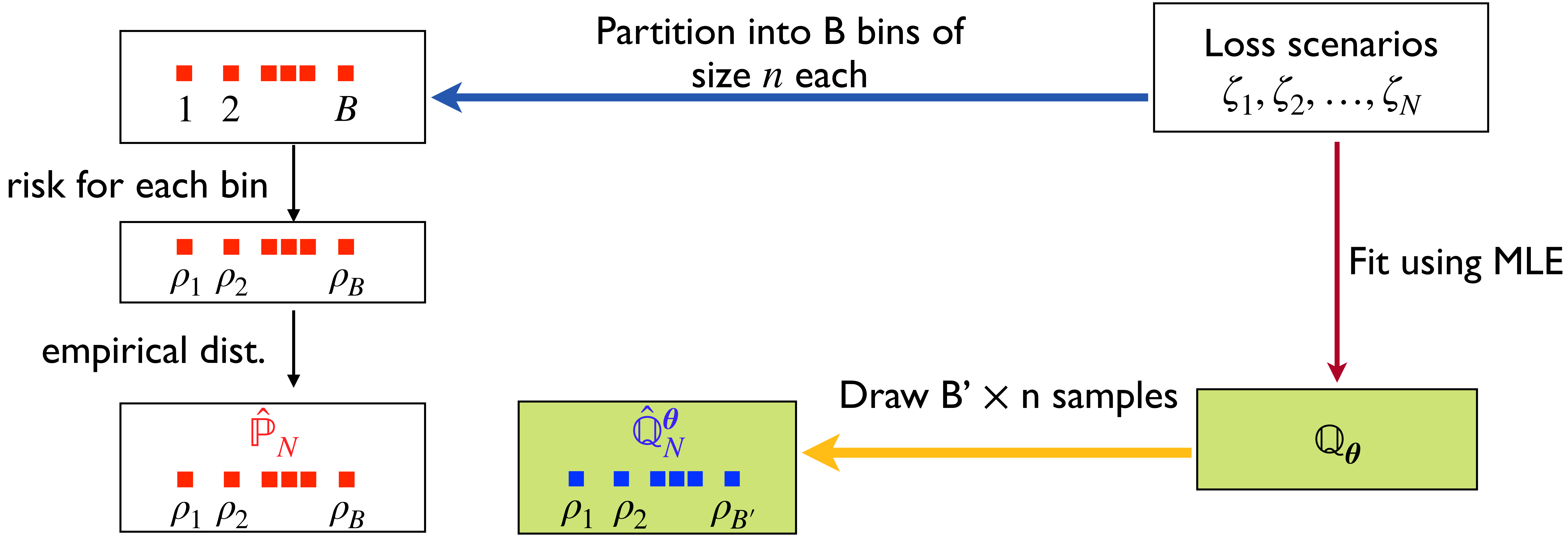

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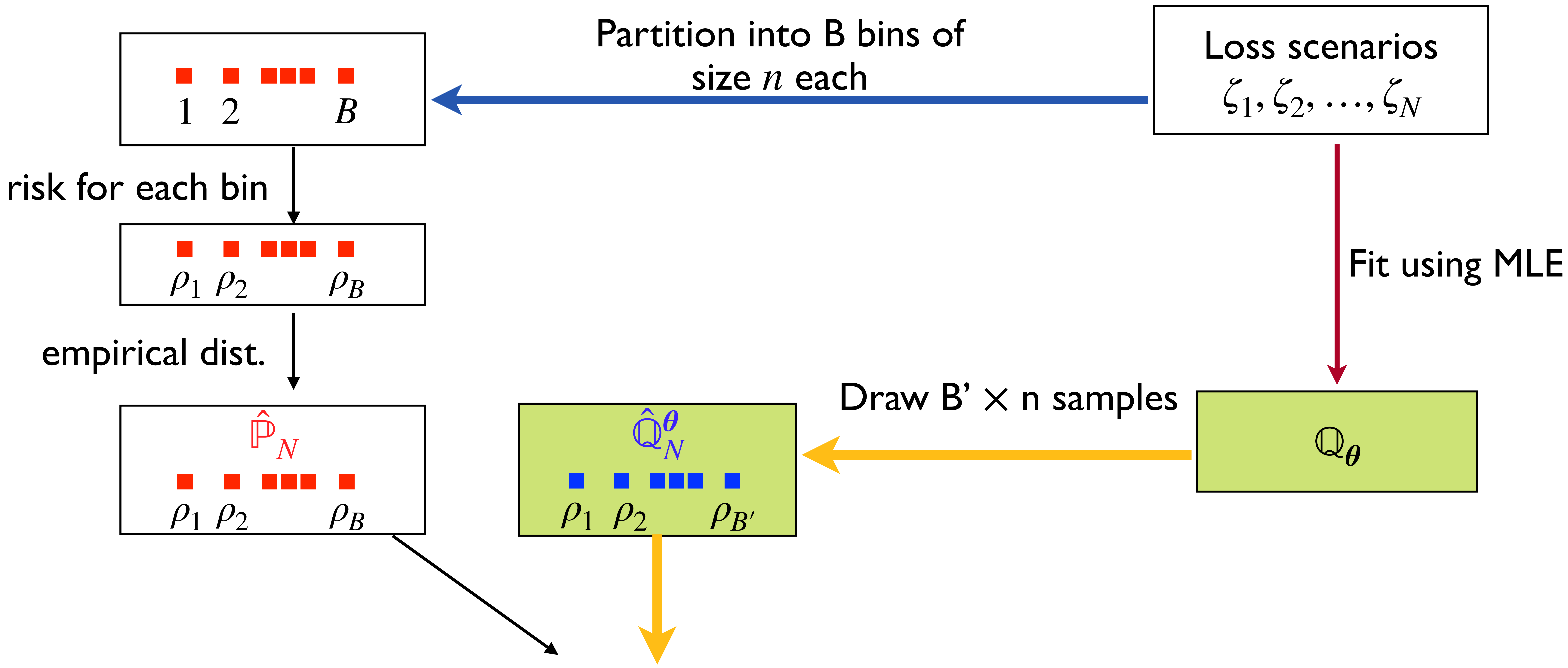

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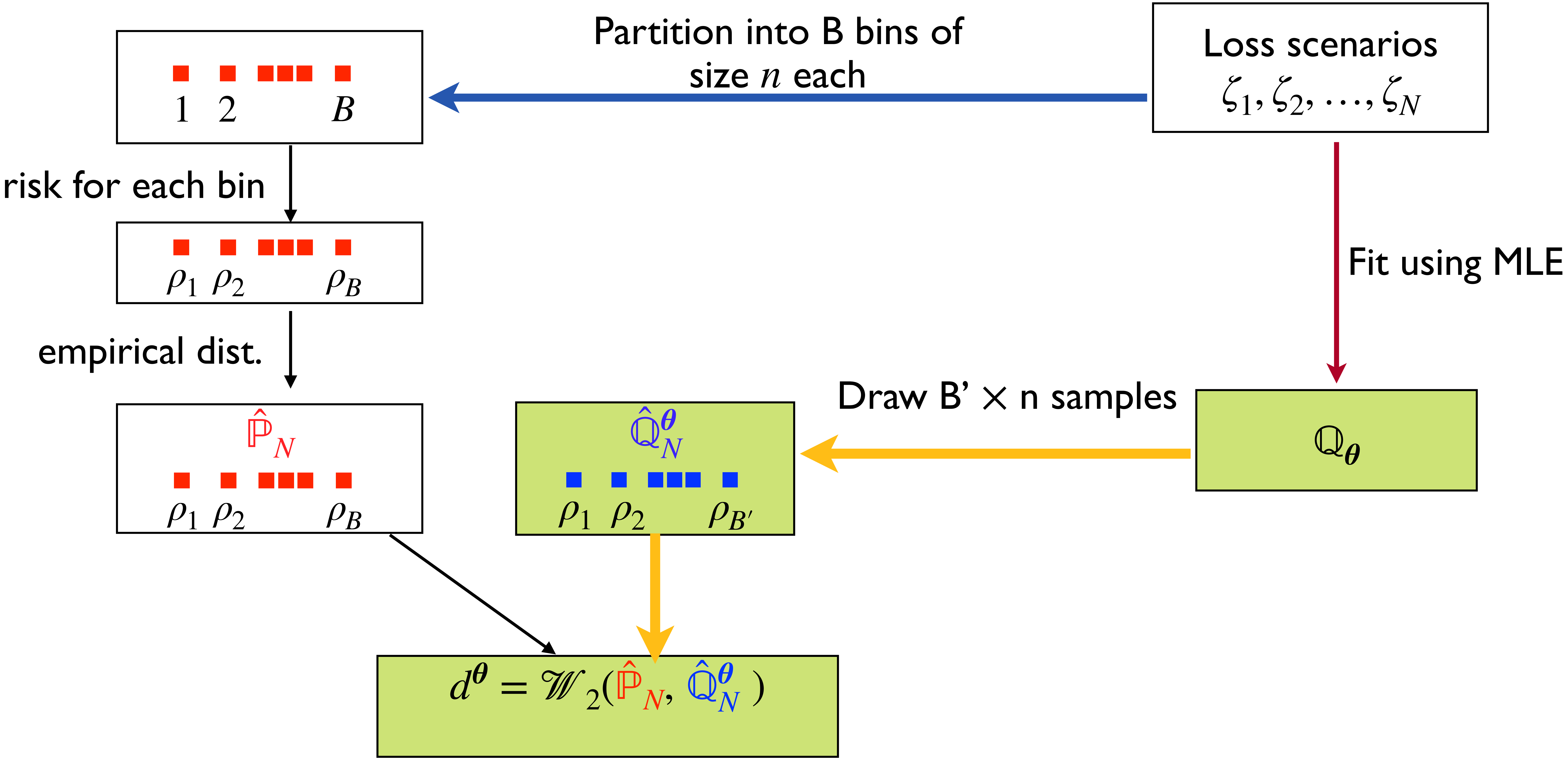

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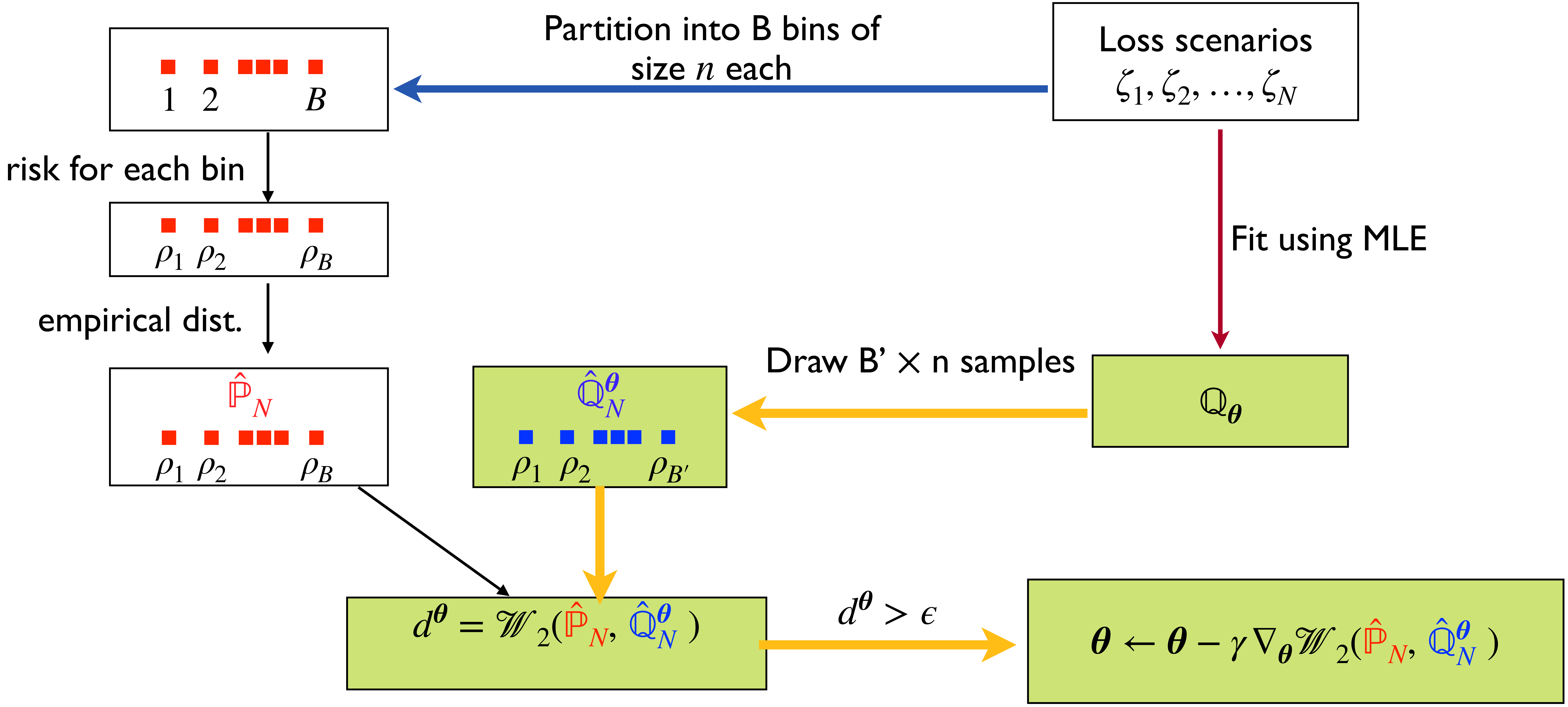

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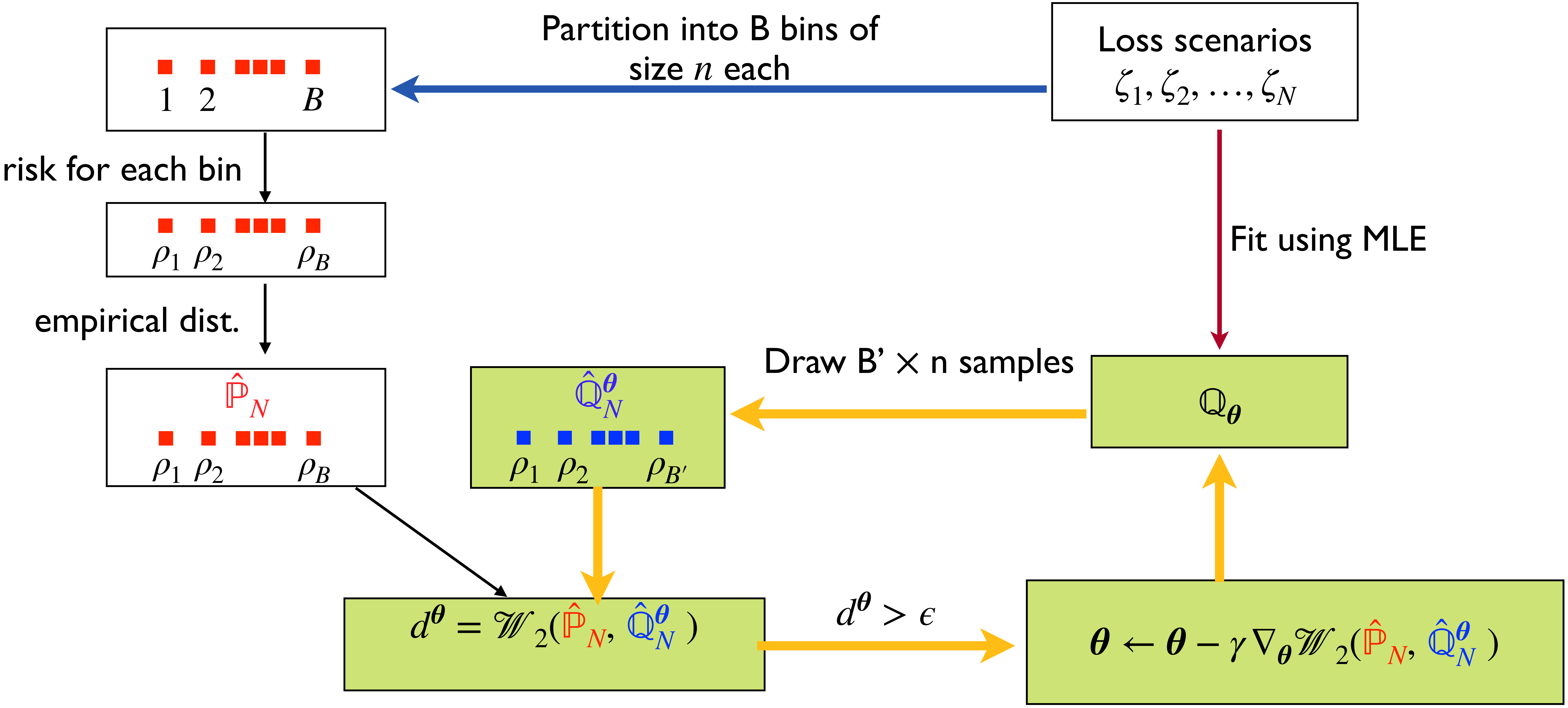

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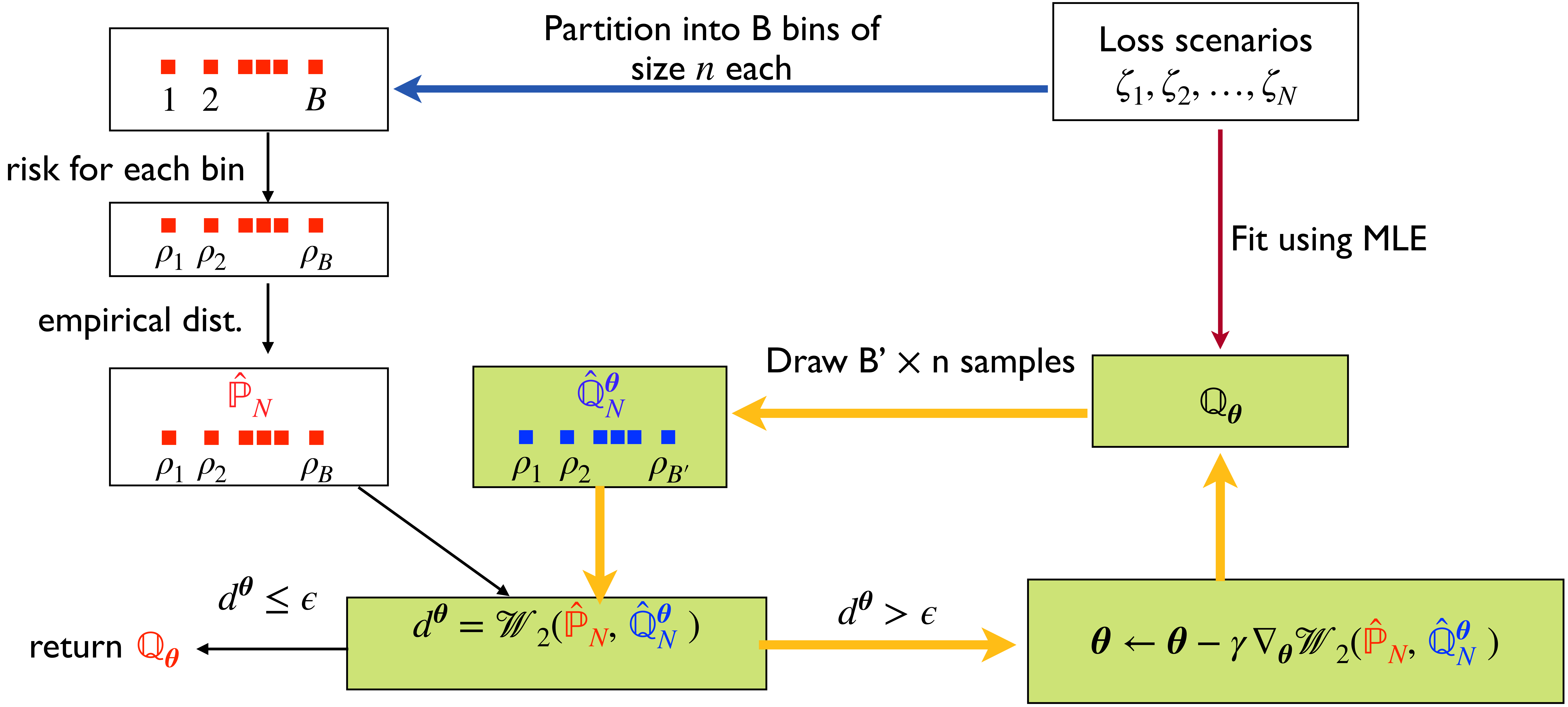

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Fisher–Tippett–Gnedenko theorem:
As $n \to \infty$, distribution of M_n converges to either Weibull, Fréchet or Gumbel

-Fit using MLE

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Our approach:

- cdf normal rv - $\Phi(\mu, \sigma)$
- Fit $\Phi^n(\mu, \sigma)$ to $m_1, m_2, \cdots, m_B$

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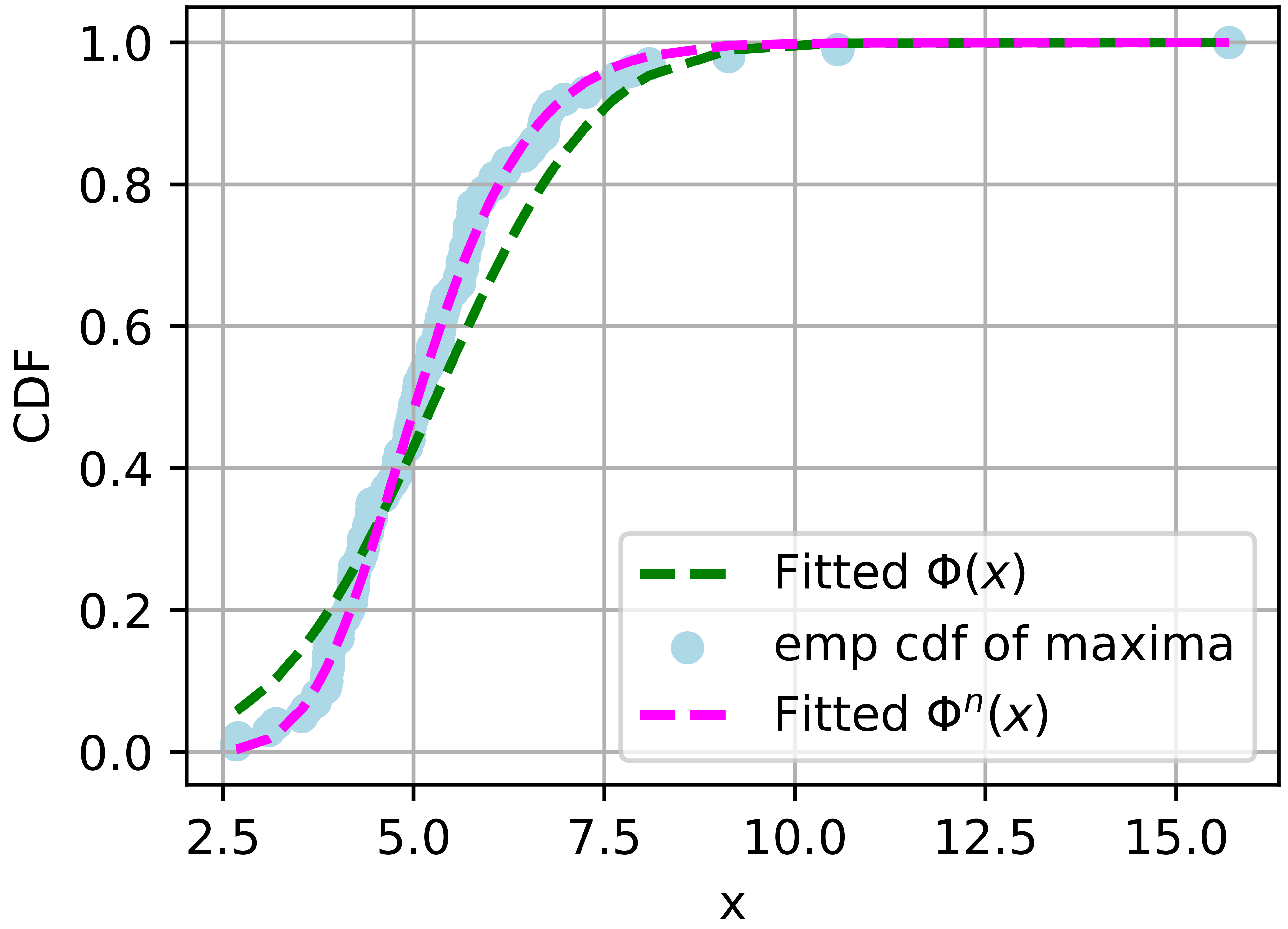

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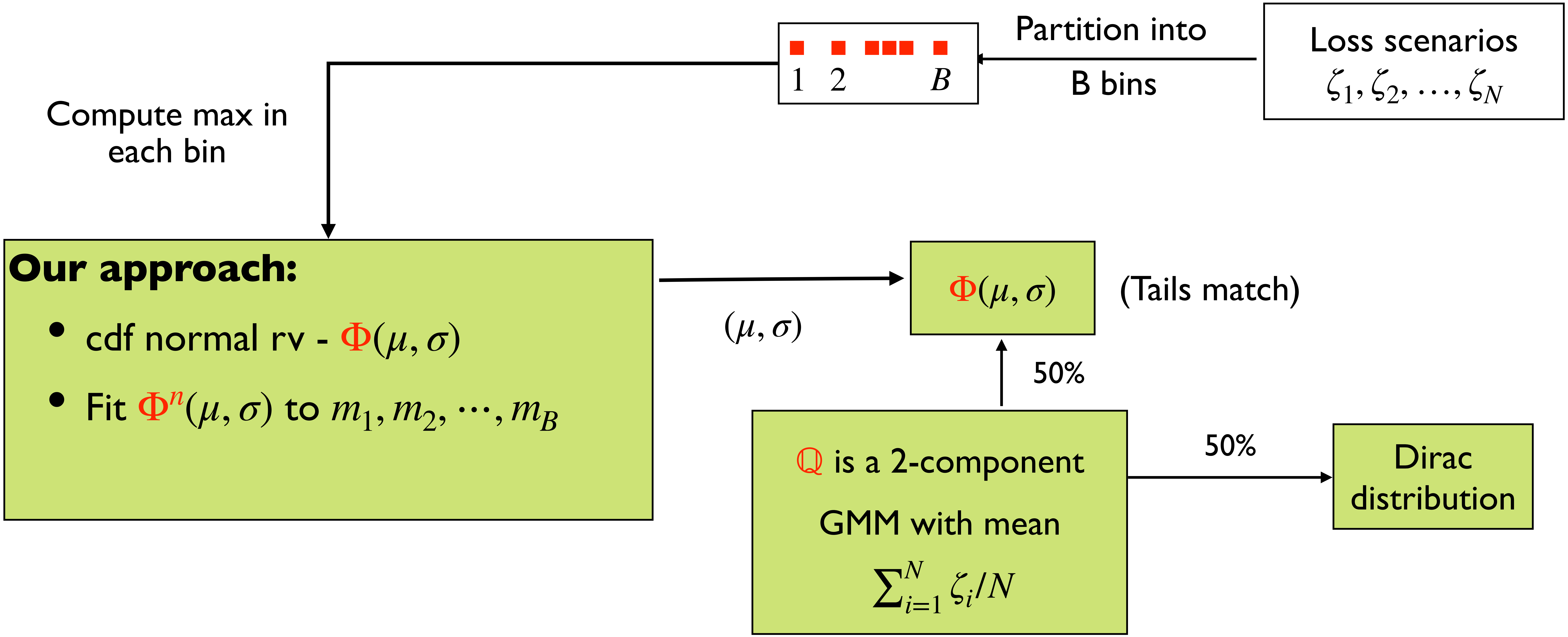
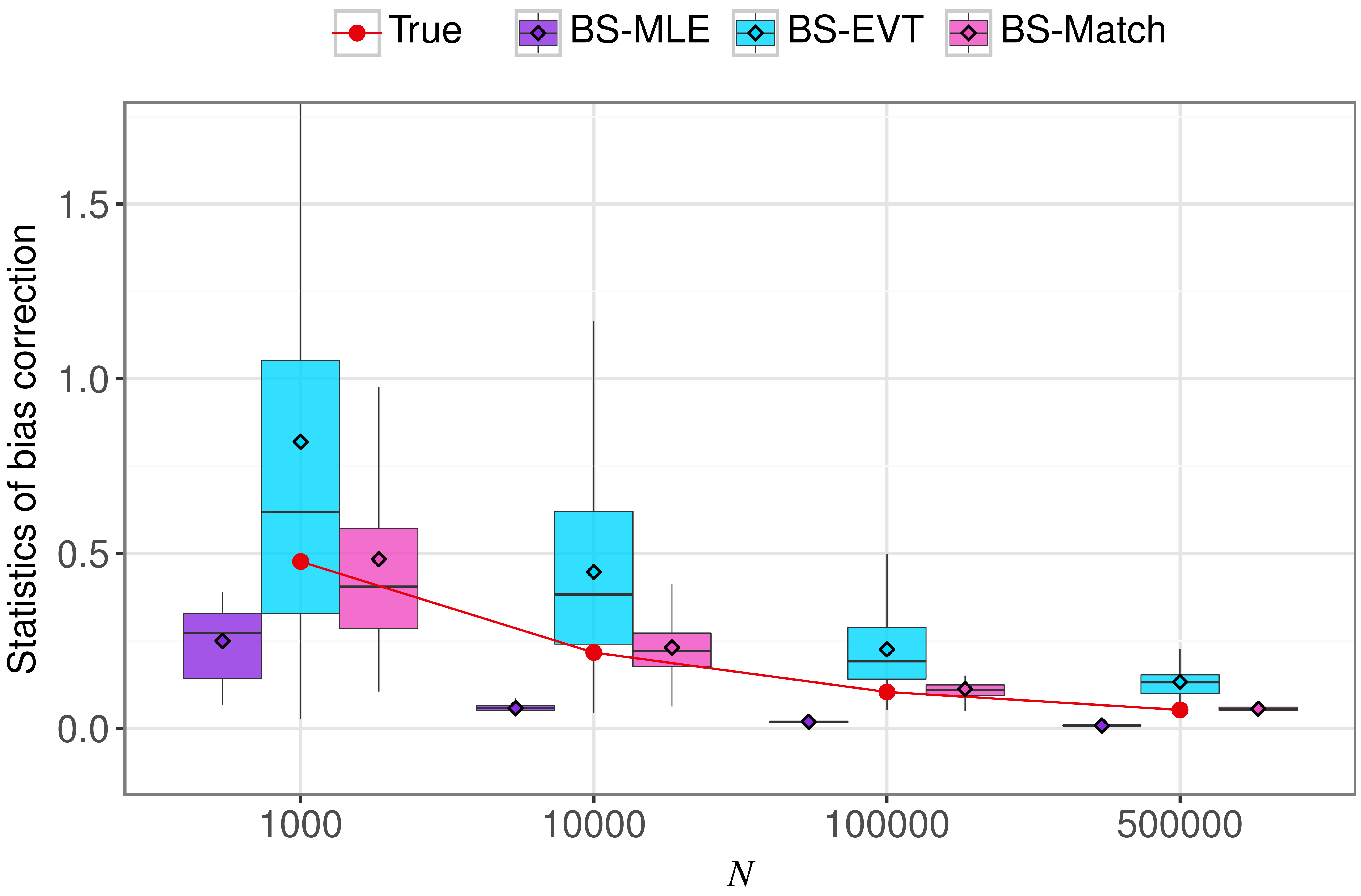


Ex 2: Bias mitigation



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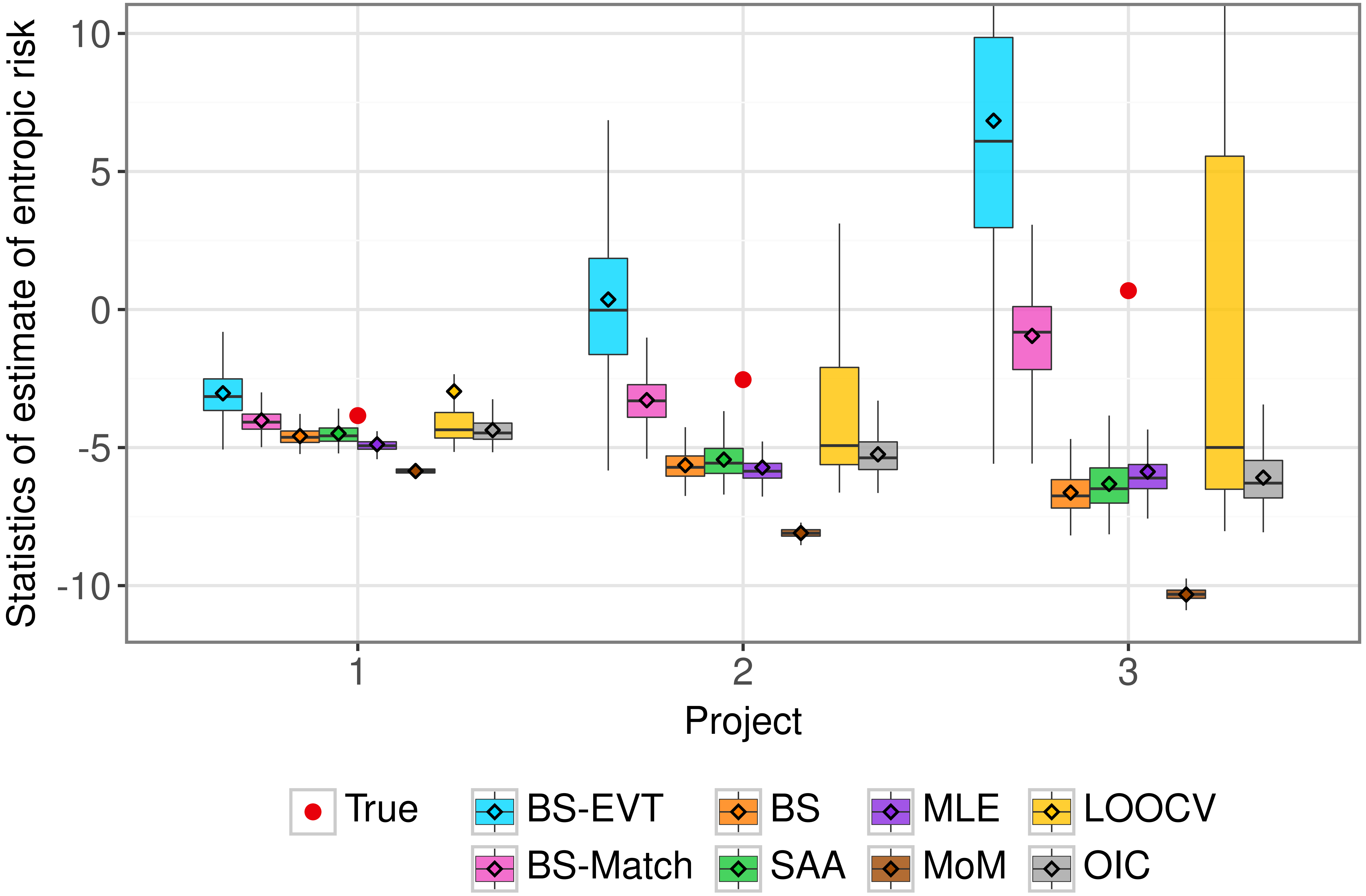
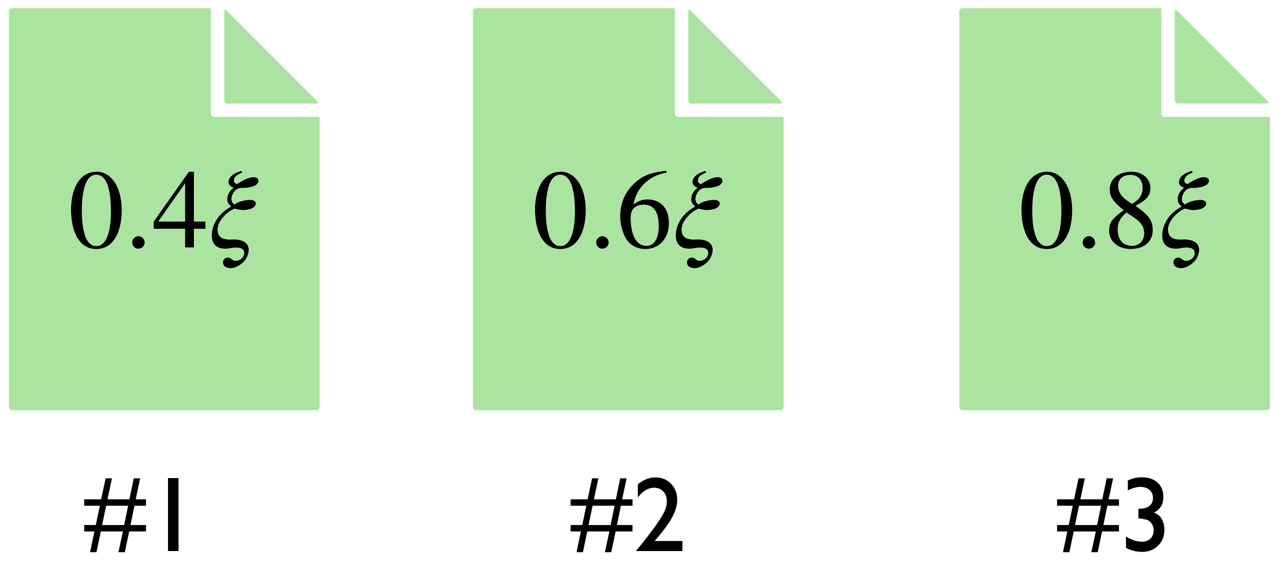
- BS-MLE - Fit $\mathbb{Q}$ using MLE
- Underestimation persists

- **BS-EVT - Fit $\mathbb{Q}$ by matching tails**
- **BS-Match - Fit $\mathbb{Q}$ by entropic risk matching**

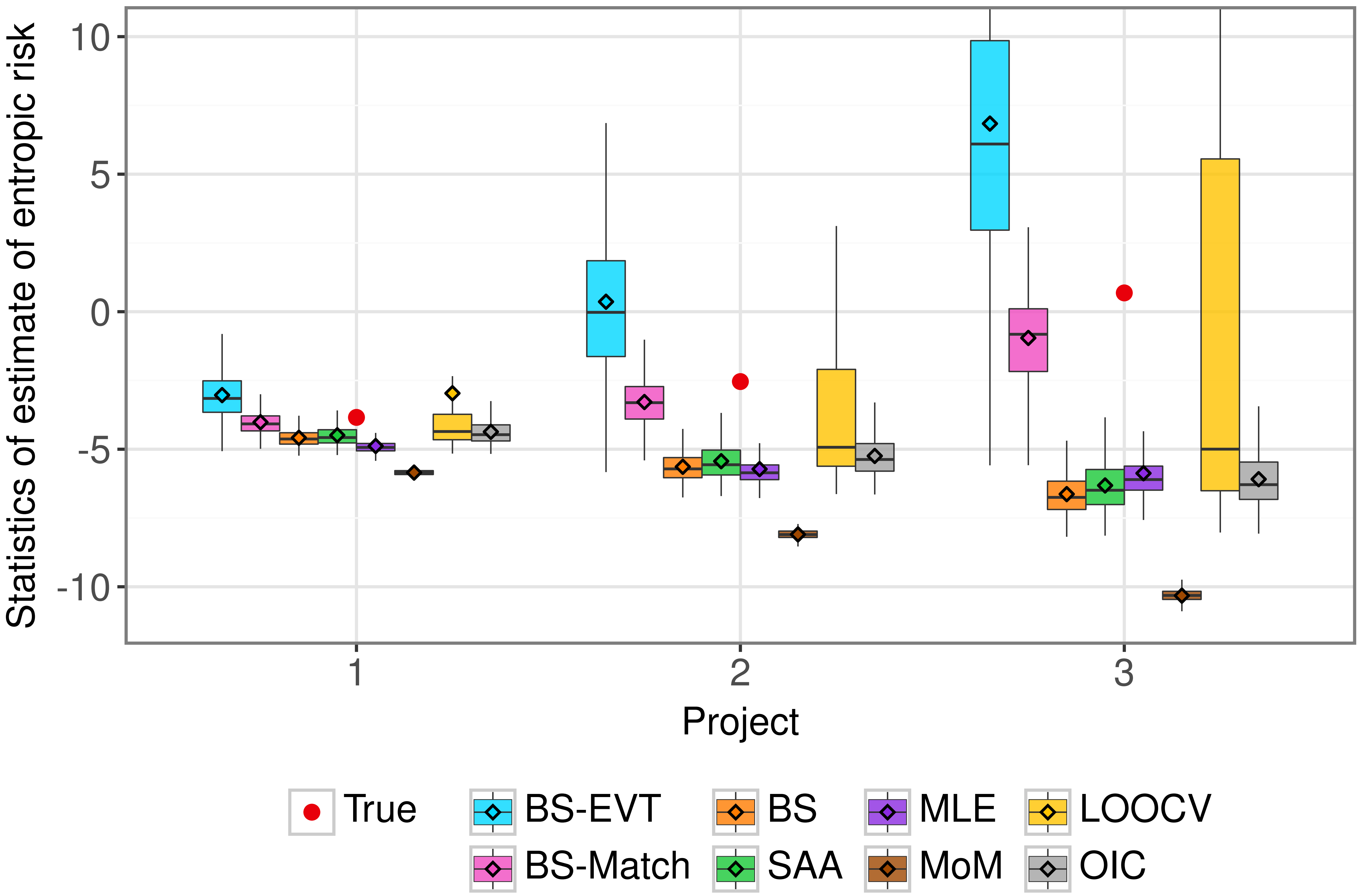
Ex3: Compare with other estimators

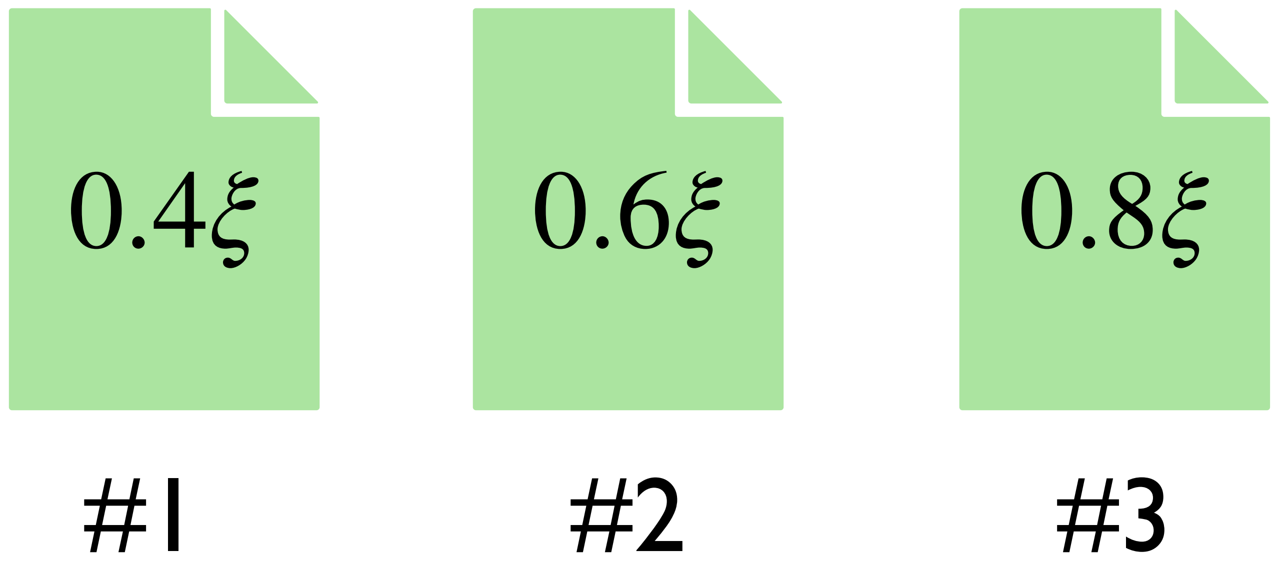




- $\xi \sim \mathrm{GMM}(\boldsymbol{\pi}, \boldsymbol{\mu}, \Sigma)$ with 5 components
- across components - $\mu_\xi = -18.6$ $\sigma_\xi = 2.9$
- Which project has lowest entropic risk based on 100 sets of 10000 samples with $\alpha = 3$?

Ex3: Compare with other estimators





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Going from estimation to optimization

Distributionally robust optimization

- Loss depends on $z \in \mathcal{Z}$:

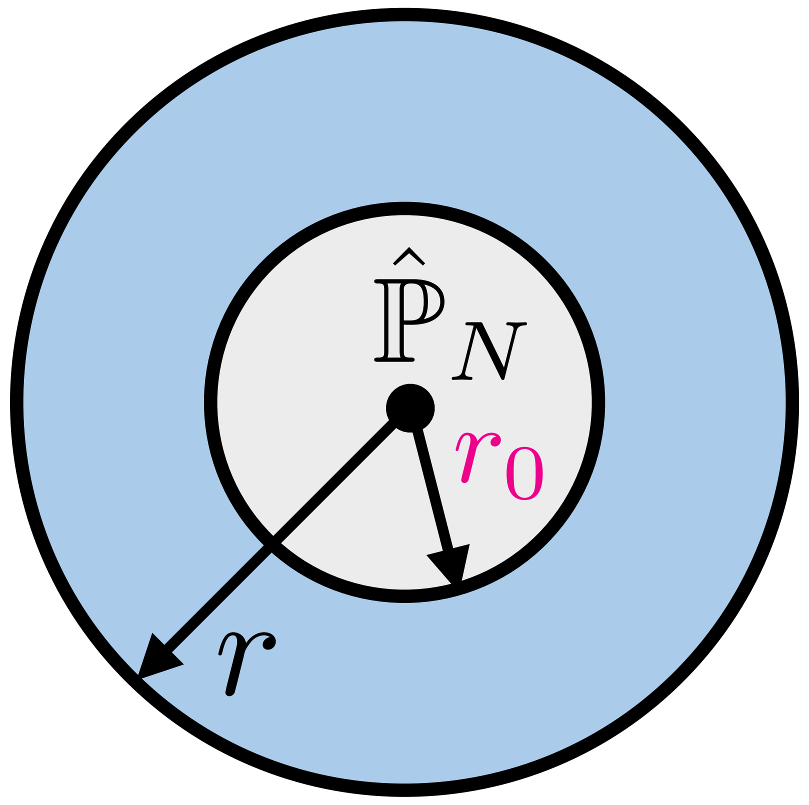
$$\rho^* = \min_{z \in \mathcal{Z}} \rho_{\mathbb{P}}(\ell(z, \boldsymbol{\xi}))$$

- Sample average approximation

$$\rho_{SAA} = \min_{z \in \mathcal{Z}} \rho_{\hat{\mathbb{P}}_N}(\ell(z, \boldsymbol{\xi}))$$

- DRO accounts for distributional ambiguity:

$$\rho_{DRO} = \min_{z \in \mathcal{Z}} \sup_{\mathbb{Q} \in \mathcal{B}_p(\epsilon)} \rho_{\mathbb{Q}}(\ell(z, \boldsymbol{\xi}))$$



$\mathcal{B}_p(\epsilon)$

Distributionally robust optimization

Distributionally robust optimization

❎ KL divergence and Type-p Wasserstein ($p < \infty$): unbounded worst-case loss

Distributionally robust optimization

- KL divergence and Type-p Wasserstein ($p < \infty$): unbounded worst-case loss
- Type $\infty-$Wasserstein: bounded loss

Distributionally robust optimization

- ☒ KL divergence and Type-p Wasserstein ($p < \infty$): unbounded worst-case loss
- ☒ Type $\infty -$ Wasserstein: bounded loss

Theorem: $\rho_{SAA} \rightarrow \rho^*$, $\rho_{DRO} \rightarrow \rho^*$ in probability at rate $\mathcal{O}(1/\sqrt{N})$

Regularized exponential cone program

Regularized exponential cone program

Theorem: With a linear loss function $\ell(z, \boldsymbol{\xi}) = z^\top \boldsymbol{\xi}$, DRO with type-$\infty$ Wasserstein ambiguity set reduces to:

$$\min_{z \in \mathcal{Z}} \frac{1}{\alpha} \log \left(\mathbb{E}_{\hat{\mathbb{P}}_N} \left[\exp(\alpha z^\top \boldsymbol{\xi}) \right] \right) + \epsilon \|z\|_*$$

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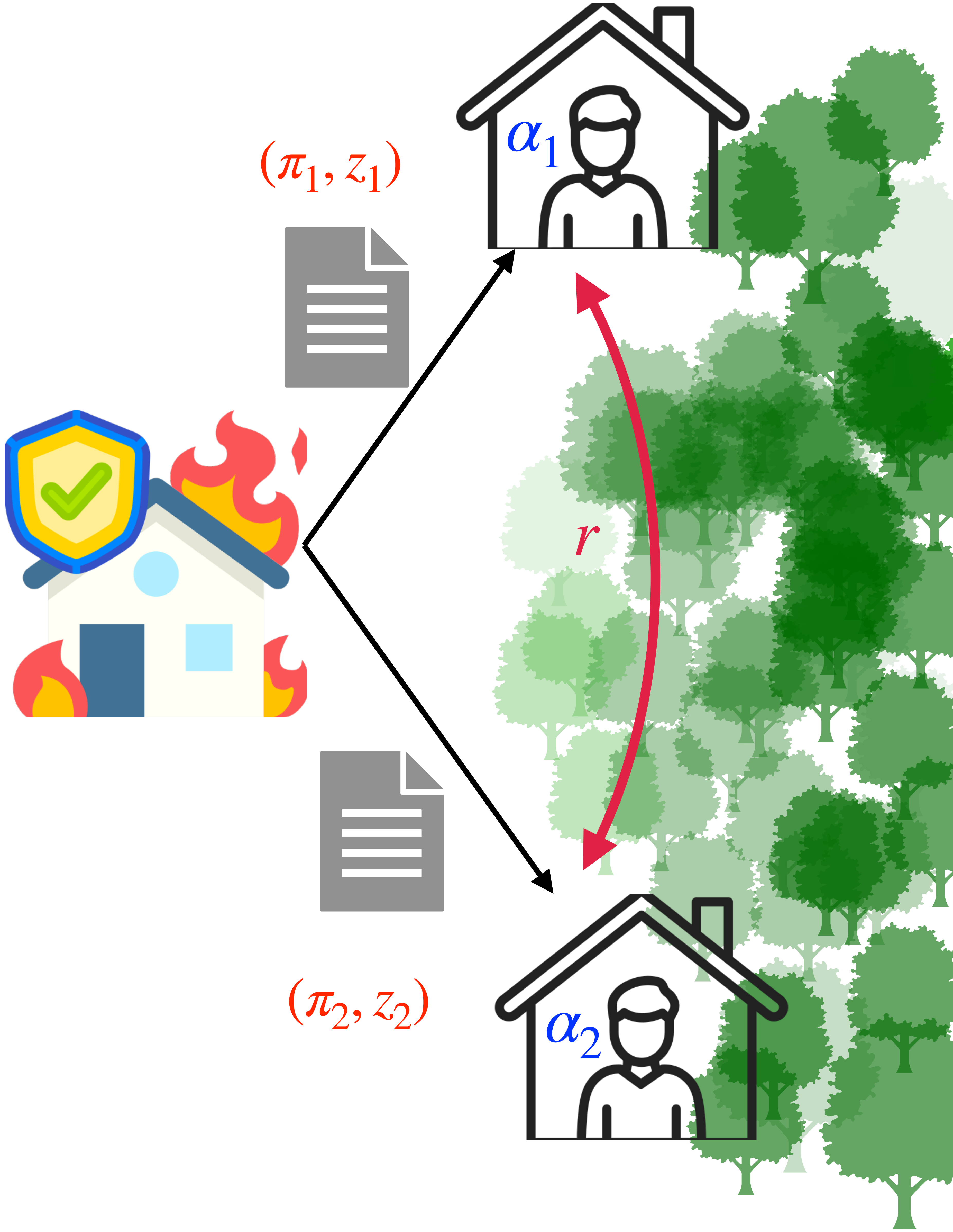
- More general loss functions - refer to our paper
- How to choose the radius ϵ?
- Validation data - underestimates the true risk
  - suboptimal radius
  - Bias correction using bootstrapping

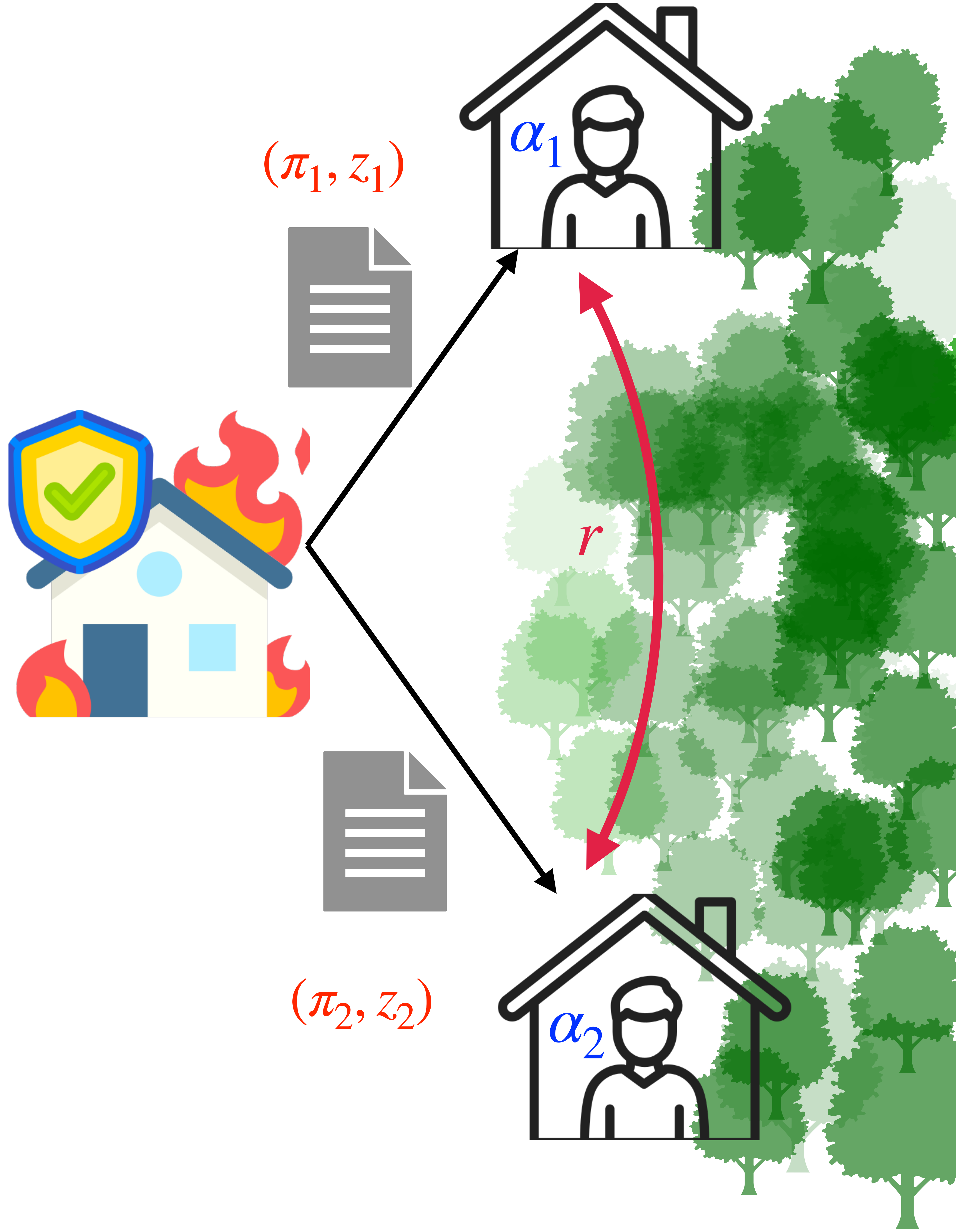
Distributionally robust insurance pricing

Distributionally robust insurance pricing



Distributionally robust insurance pricing

- Insurer offers coverage $z_h \xi$ at premium π_h
- α_h: homeowner's risk preference
- α_0: insurer's risk preference



Distributionally robust insurance pricing

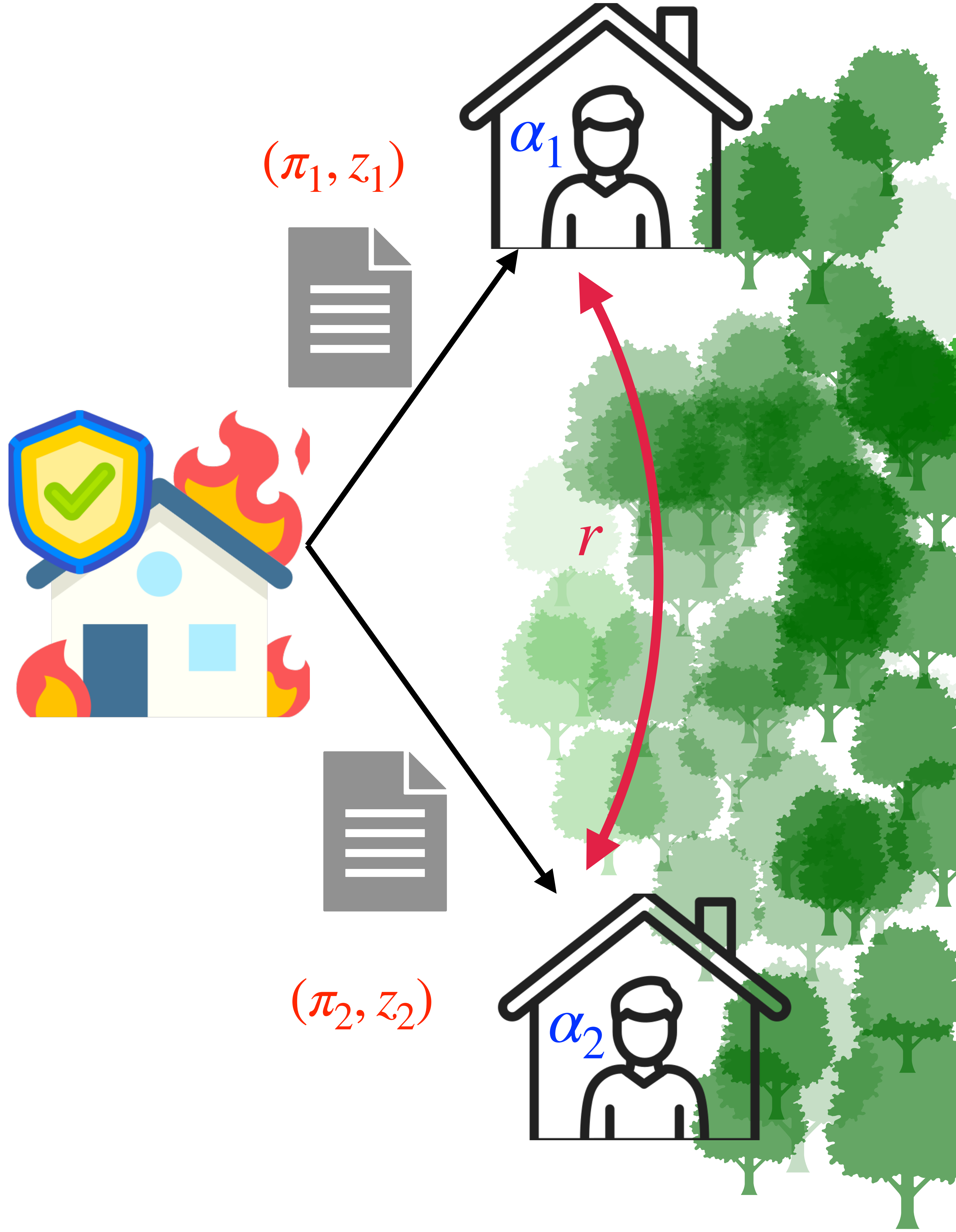
- Insurer offers coverage $z_h\xi$ at premium π_h
- α_h: homeowner's risk preference
- α_0: insurer's risk preference

$$\min \sup_{\mathbb{Q}\in\mathscr{B}_\infty(\epsilon)} \rho_{\mathbb{Q}}^{\alpha_0}\left(\boldsymbol{z}^\top\boldsymbol{\xi} - \mathbf{1}^\top\boldsymbol{\pi}\right) + \epsilon\|\boldsymbol{z}\|_*$$

$$\text{s.t.} \quad \boldsymbol{\pi} \in \mathbb{R}_+^M, \boldsymbol{z} \in [0,1]^M$$

$$\boxed{\rho_{\hat{\mathbb{P}}_{h,N}}^{\alpha_h}\left(\pi_h + (1 - z_h)\xi_h\right) \leq \rho_{\hat{\mathbb{P}}_{h,N}}^{\alpha_h}\left(\xi_h\right) \; \forall h}$$

Demand response model: Household accept/reject the contract based on their estimate of empirical entropic risk



Reformulation as exponential cone

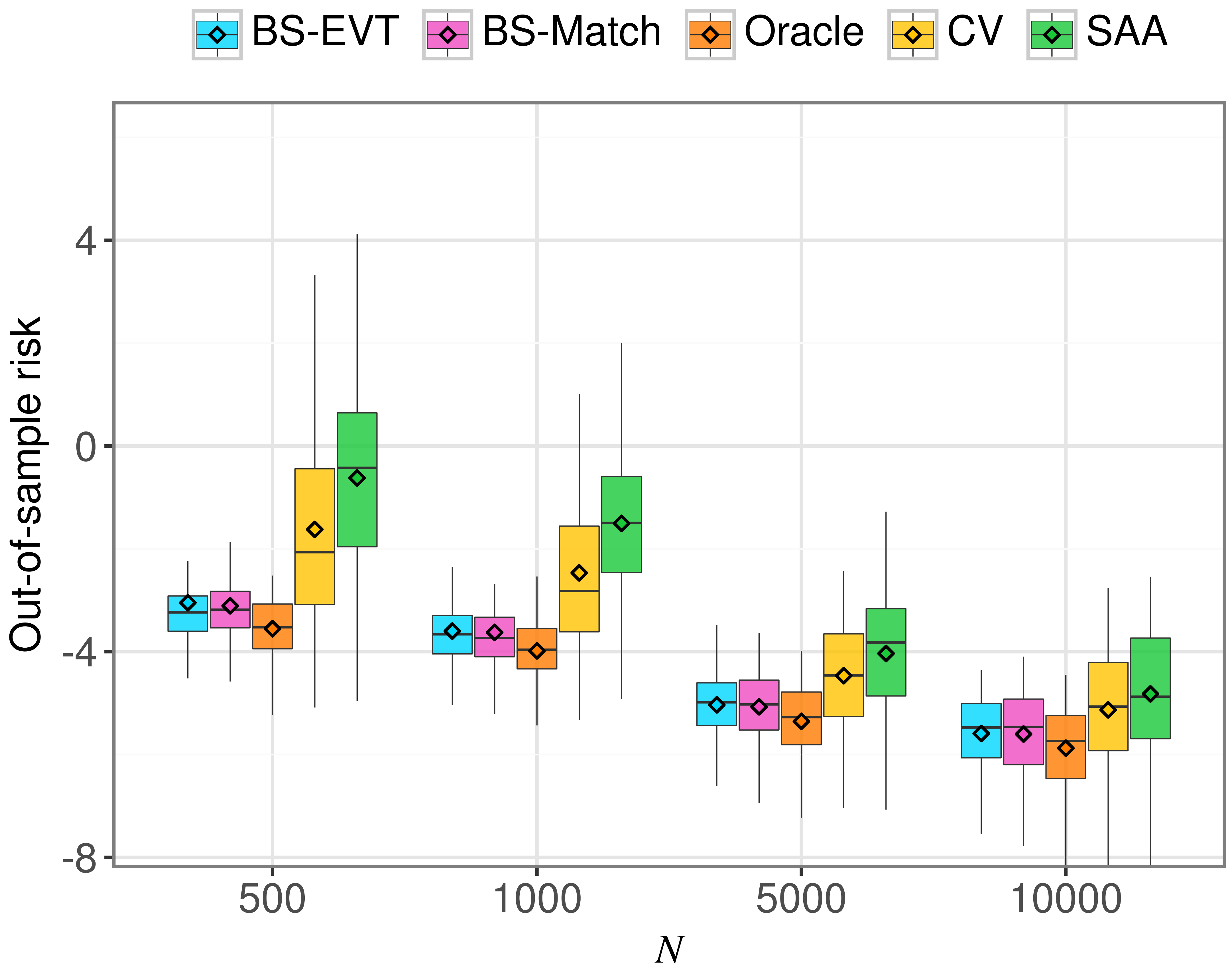
- A coverage of $z_h \xi$ is offered at premium π_h
- α_h: homeowner's risk preference
- α_0: insurer's risk preference

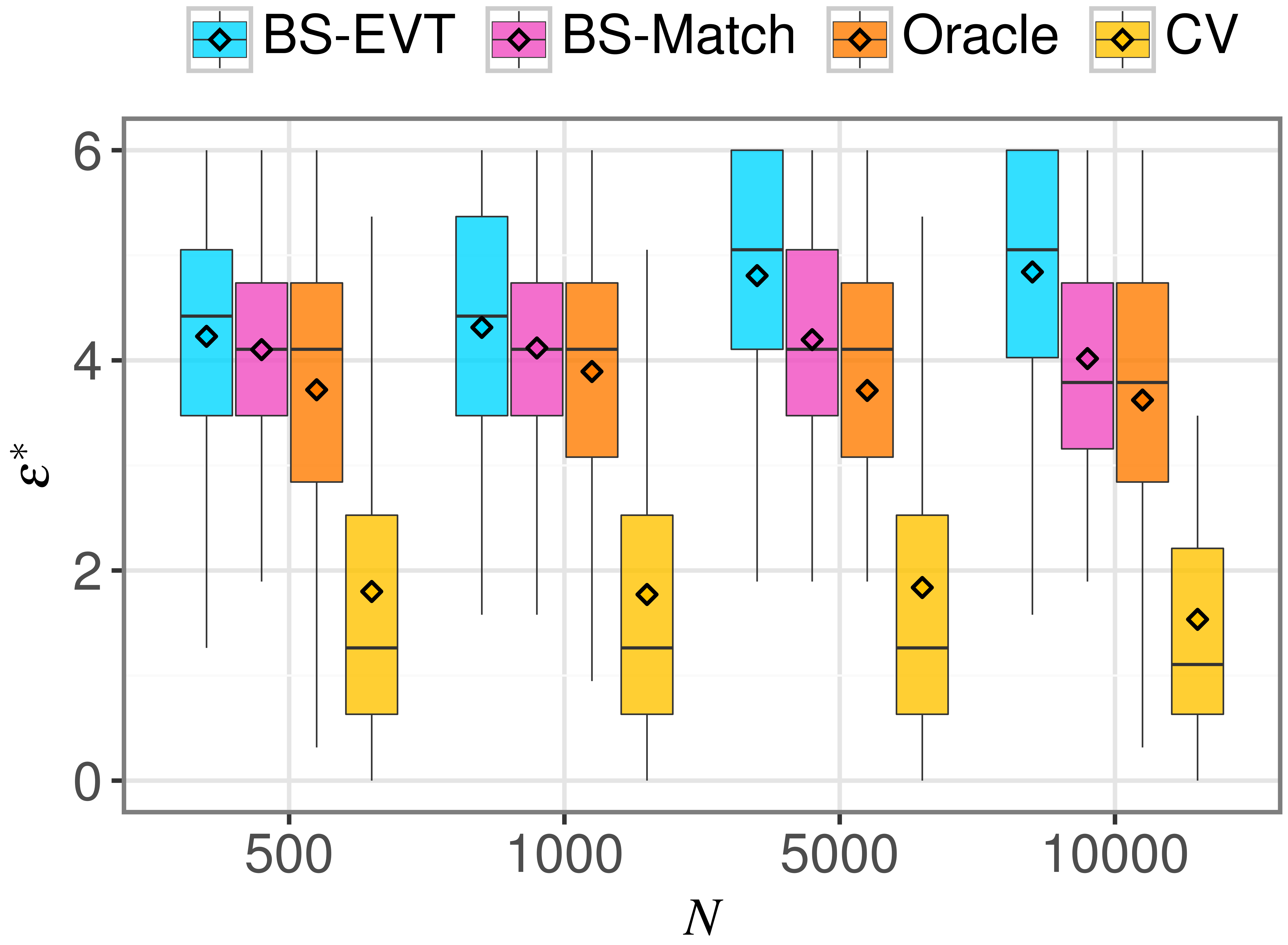
$$
\begin{aligned}
\min \quad & \rho^{\alpha_0}_{\hat{\mathbb{P}}_N}\left(\boldsymbol{z}^\top \boldsymbol{\xi} - \mathbf{1}^\top \boldsymbol{\pi}\right) + \epsilon \|\boldsymbol{z}\|_* \\
\text{s.t.} \quad & \boldsymbol{\pi} \in \mathbb{R}^M_+, \boldsymbol{z} \in [0,1]^M \\
& \rho^{\alpha_h}_{\hat{\mathbb{P}}_{h,N}}\left(\pi_h + (1 - z_h)\xi_h\right) \le \rho^{\alpha_h}_{\hat{\mathbb{P}}_{h,N}}\left(\xi_h\right) \ \forall h
\end{aligned}
$$

Data for numerical experiments:

Loss scenarios are generated from Gaussian copula with $\Gamma(\kappa_h, \lambda_h)$ marginals

Out-of-sample risk and radius - vary N

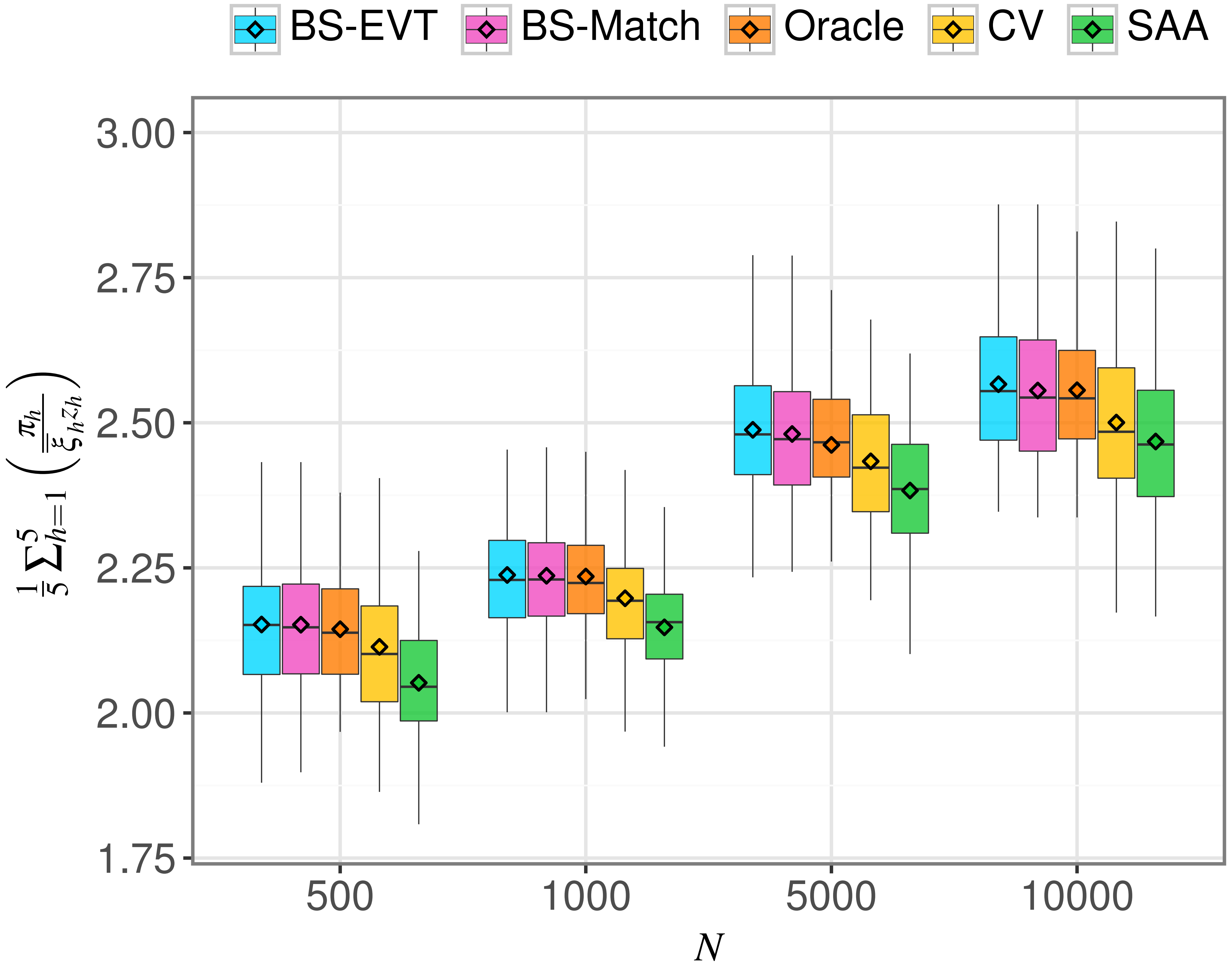




Risk decreases as training samples increase

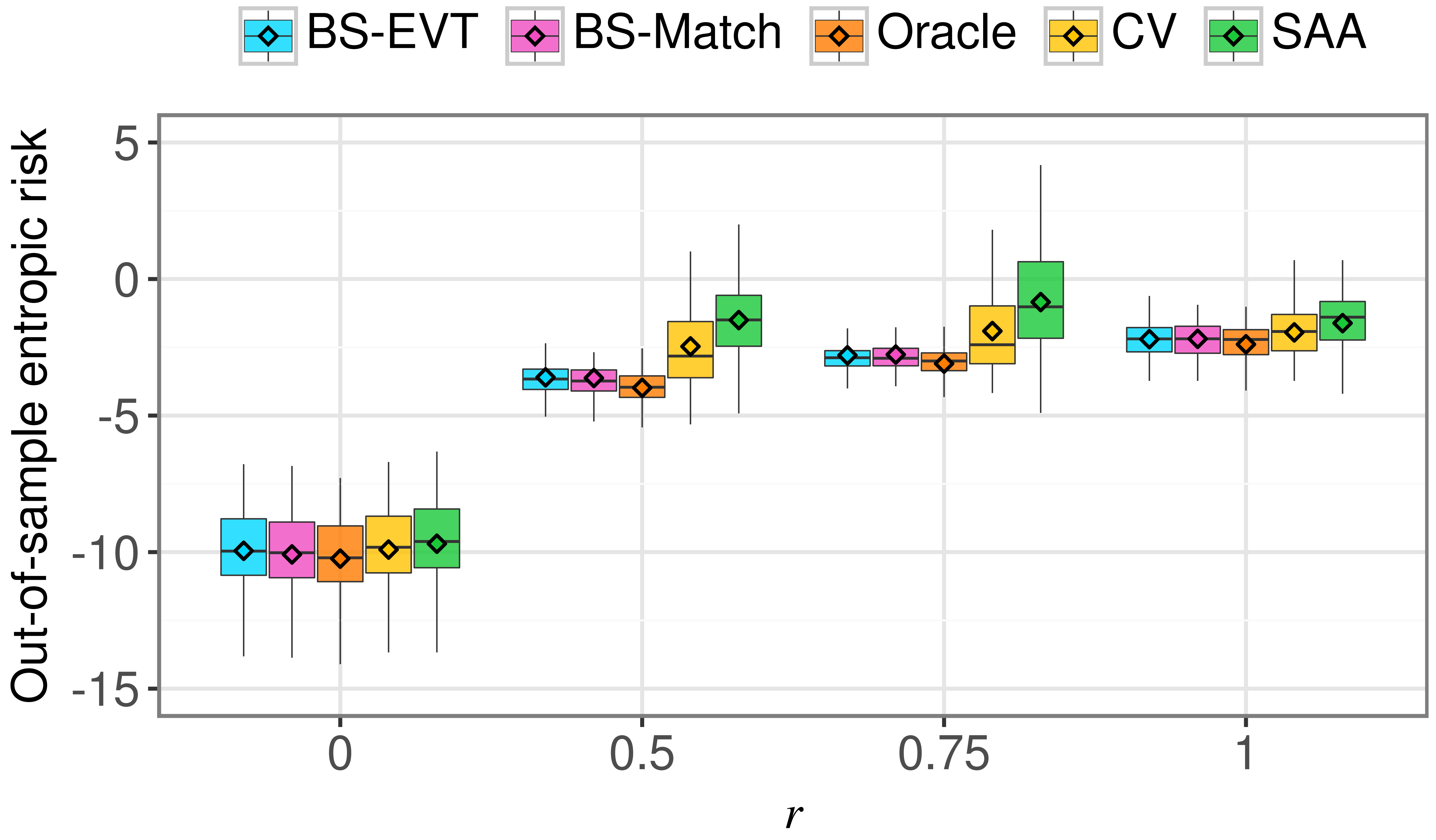
Our models choose higher radius while traditional CV chooses lower radius

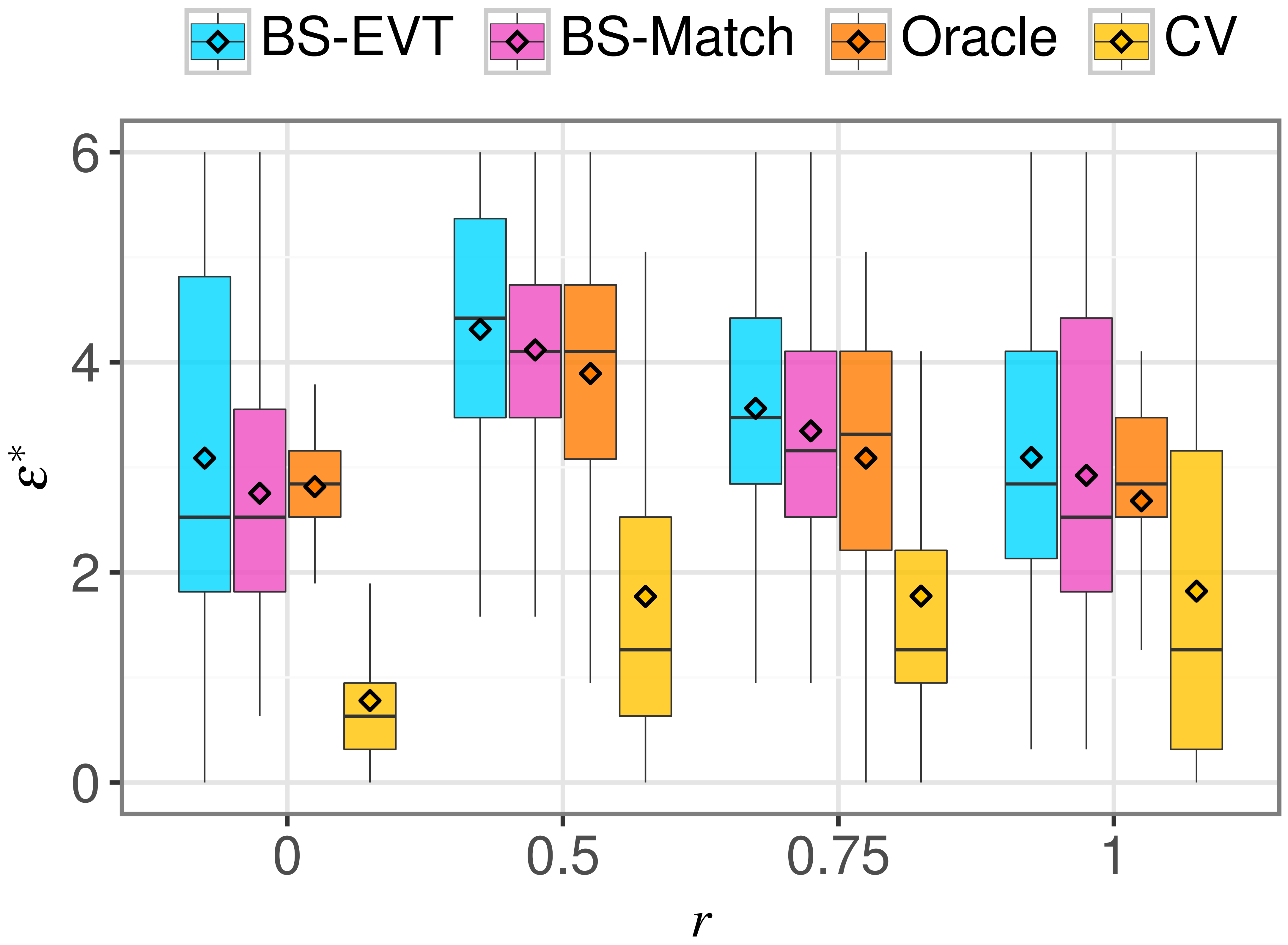
Premium per unit coverage - vary N



Households pay higher premiums as their estimates of risk improve with N

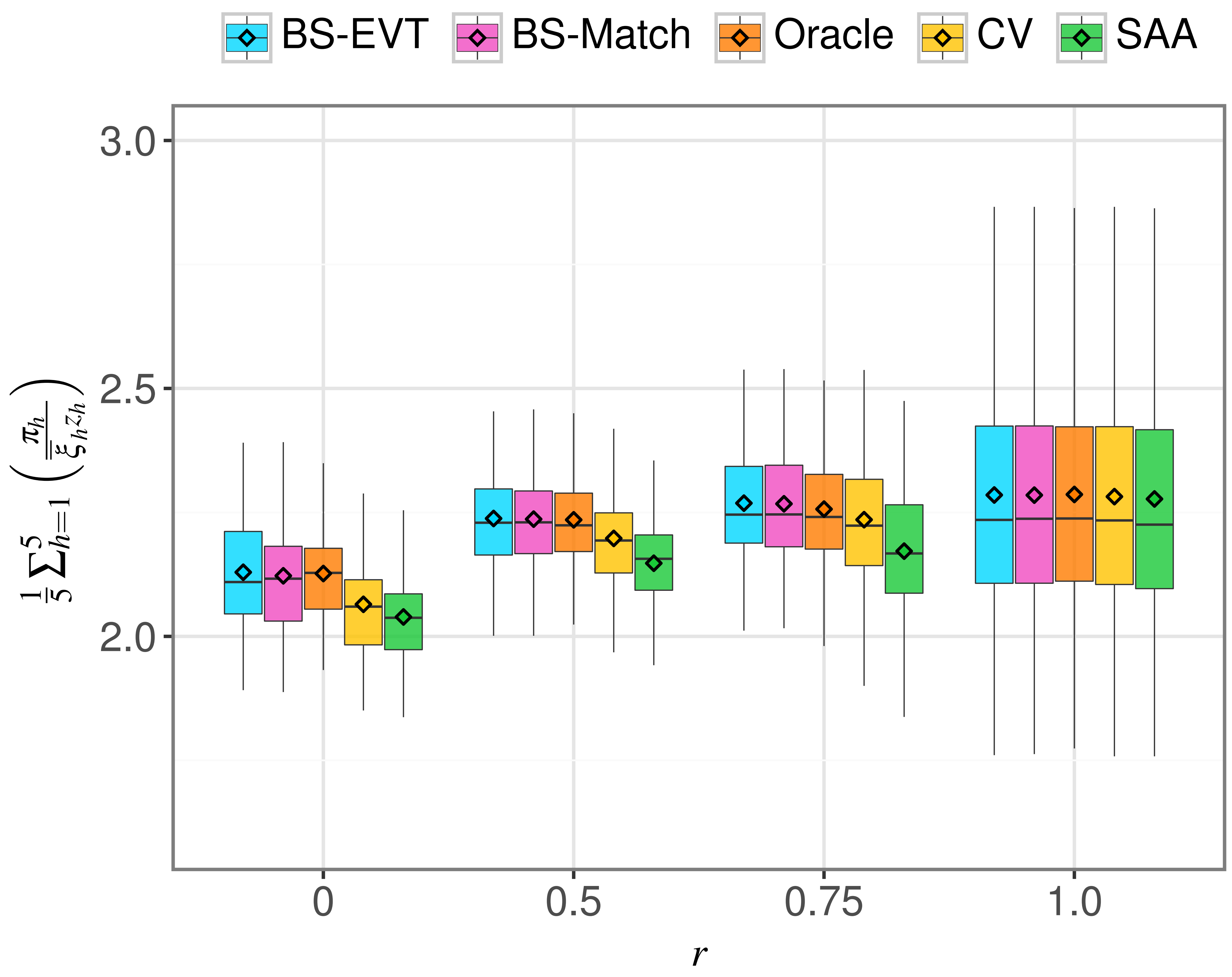
Out-of-sample risk and radius - vary correlation





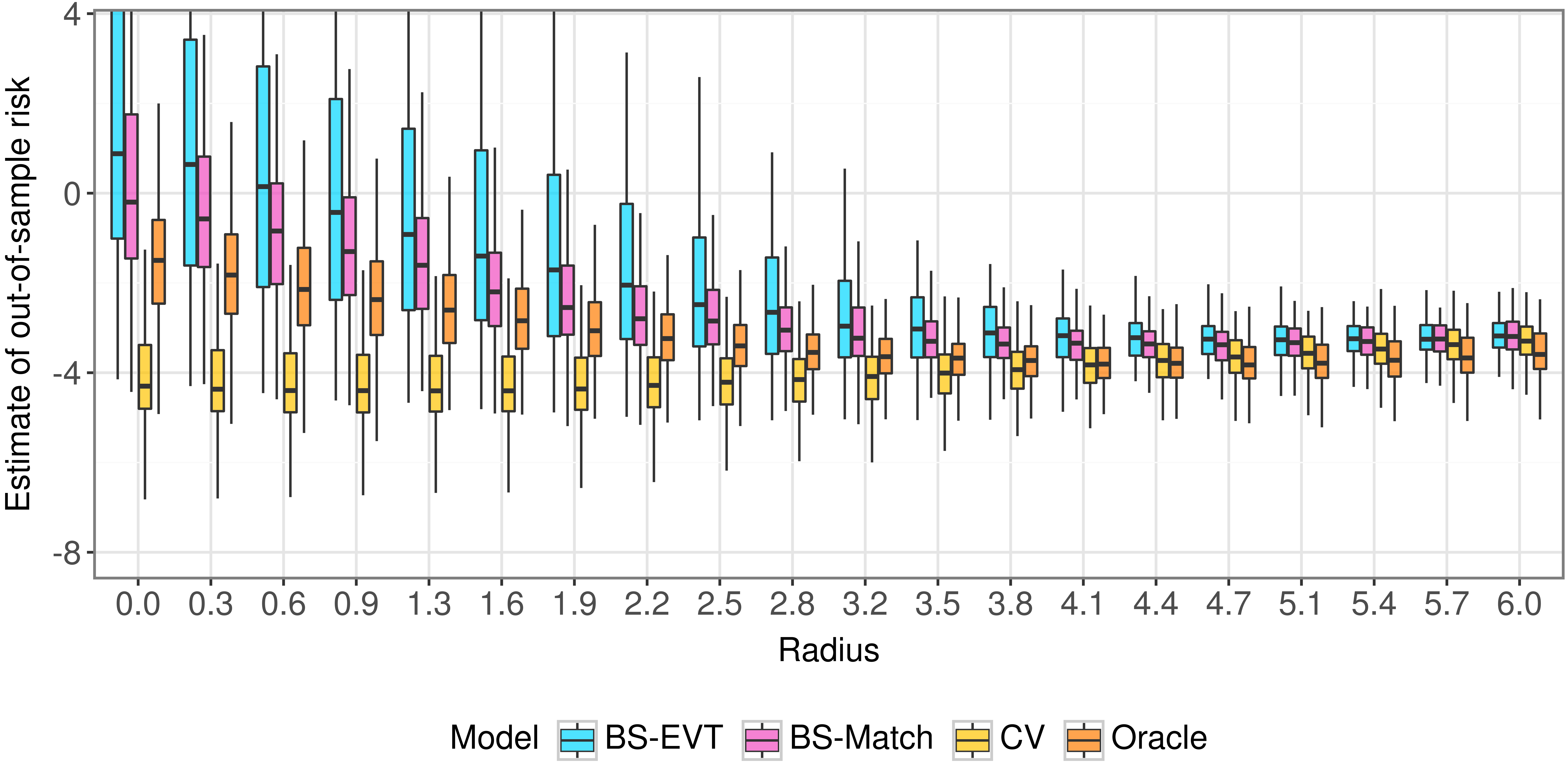
High correlation: extreme loss events more likely to occur simultaneously, increasing insurer's risk exposure

Premium per unit coverage - vary correlation



High correlation: benefits of risk pooling diminish, reduce coverage significantly to reduce risk exposure

Why our models identify better radius?



Take-away message

- Entropic risk estimation and optimization
  - Two practical approaches to reduce optimistic bias
- Future research:
  - Extend to CVaR
  - Solve exponential cones faster

Link to paper

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